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EXISTENCE OF SOLUTIONS OF STOCHASTIC NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAY

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ABSTRACT:

In this paper we study the existence of mild solutions for a class of stochastic neutral functional differential equations with state dependent delay in an abstract space by using the fixed point theorem of Leray-Schauder alternative.

Key words:*Existence of solutions, Leray-Schauder fixed point theorem, Stochastic differential equations, State dependent delay.*

AMS(MOS) ***Subject Classifications*:34K40, 34A60, 34K50, 60H10, 93E03.**

1. INTRODUCTION

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems. Neutral functional differential systems arise in many areas of applied mathematics and for this reason these systems have been extensively investigated, refer [6] in the last few decades. There are many contributions relative to this topic and we refer readers to [9, 7] and the monograph [8, 5]. Recently, much attention has been paid to existence results for partial functional differential equations with state-dependent delay, and we cite the works [17, 15, 16, 21] and similarly with unbounded delays and infinite delays we refer [18, 7, 11, 25]and monograph [4] and the references therein.The global existence results for functional integro-differential stochastic evolution equations in Hilbert space have been studied elaborately in [13]. Also Taniguchi et al. [11] established the unique solution of stochastic functional differential equations in Hilbert space using the contraction mapping principle. The existence results of differential equations have been generalized to stochastic functional differential inclusions see [10, 14, 24] using the fixed point argument.

The theory of stochastic differential equations has become an active area of research in recent years due to their numerous applications with non-deterministic nature for problems arising in mechanics, electrical engineering, medicine biology and other areas of science. In the literature, there are very few results concerning stochastic non-linear systems see [22, 23, 27] and references therein. The above works mainly deal with existence and controllability investigation for impulsive stochastic systems and stability of sobolev-type equations. To the best of authors' knowledge, the problem of existence of solutions of stochastic neutral functional differential equations with state dependent delays have not been fully investigated in the literature.

Motivated by the above discussions, the results presented in the current manuscript constitute a continuation and generalization of existence, uniqueness and controllability results from [10, 14, 22, 20, 11] in two ways. For one, we study the class of neutral stochastic functional differential equations with state dependent delay using an abstract phase space, together with the Leray-Schauder fixed point theorem. To the authors knowledge, this approach has not yet been applied in the study of such stochastic problems in the literature. And two, our result constitute a stochastic variant of the results concerning the existence of mild solutions in [16]; this enables one to introduce noise into the concrete models that are subsumed as special cases of the abstract system with state dependent delay being studied, thereby allowing for a more accurate description of the phenomenon.

In this paper, we are interested to study the existence of solutions of the following nonlinear neutral stochastic functional differential equation with state dependent delay in a Hilbert space,

$$\begin{cases} d[x(t)+F(t,x_t)] & = \Big[Ax(t)+h(t,x_t)\Big]dt + G(t,x_{\rho(t,x_t)})dW(t), \quad \text{for a.e } t\in J:=[0,a], 0<a<\infty, \\ x(t) & = \phi(t)\in \mathcal{L}_2^0(\Omega,\mathcal{B}),\ a.e \quad t\in J_0=(-\infty,0] \end{cases} \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of closed linear operator $S(t), t\geq 0$, on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. $x:[-r,a]\to H$ is the initial datum such that $\phi(t)$ is the $\mathfrak{F}_0$-measureable for all $t\in J_0, E|\phi(0)|^2<\infty$ and $\int_{-r}^{0} E|\phi(s)|^2 ds<\infty$. Let K be another separable Hilbert space with the inner product $(\cdot,\cdot)_K$ and norm $\|.\|_K$. Suppose $W(t)$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$. Let $L(K,H)$ denote the Banach space of all bounded linear operators from K into H. The histories x_t belongs to some abstract phase space $\mathcal{B}$ defined axiomatically (see Section 2); $F,h: J\times\mathcal{B}\to H$ are the measurable mappings in H- norm and $G : J \times J \times \mathcal{B} \to L_Q(K,H)$, ($L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H which is going to be defined below) is a measurable mappings in $L_Q(K,H)$-norm. $\phi(t)$ is a $\mathcal{B}$-valued random variable independent of Brownian motion $W(t)$ with finite second moment and $\rho : J \times \mathcal{B} \to \mathbb{R}$ is an appropriate function.

This paper is organised as follows: In section 2, we give some basic notations and some preliminary lemmas which are more essential to this paper. In section 3, we discuss the main result. Finally, conclusion is given in section 4.

2. PRELIMINARIES

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t, t \in I\}$ satisfying $\mathfrak{F}_t \subset \mathfrak{F}$. An *$H$-valued random variable* is an $\mathfrak{F}$-measurable function $x(t) : \Omega \to H$, and a collection of random variables

$$S = \{x(t,\omega) : \Omega \to H | t \in I\}$$

is called a *stochastic process*. Usually, we suppress the dependence on $\omega \in \Omega$ and write $x(t)$ instead of $x(t,\omega)$ and $x(t) : I \to H$ in the place of S. Let $\beta_n(t)(n = 1, 2, ...)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathfrak{F}, P)$. Set

$$\beta(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) \zeta_n, \ t \geq 0,$$

where $\lambda_n \geq 0$, (n=1, 2, ...) are nonnegative real numbers and $\{\zeta_n\}$ (n=1, 2, ...) is complete orthonormal basis in K. Let $Q \in L(K,K)$ be an operator defined by $Q\zeta_n = \lambda_n \zeta_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $\beta(t)$ is called a Q-Wiener process. We assume that $\mathfrak{F}_t = \sigma(\beta(s) : 0 \leq s \leq t)$ is the σ-algebra generated by β and $\mathfrak{F}_T = \mathfrak{F}$. Let $\varphi \in L(K,H)$ and define

$$\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \varphi \zeta_n\|^2.$$

If $\|\varphi\|_Q < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\varphi : K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\varphi\|_Q = \langle\langle\varphi, \varphi\rangle\rangle^{1/2}$ is a Hilbert space with the above norm topology.

Finally, let $C(I, L_2(\Omega, H))$ stands for the space of all continuous functions φ from I into $L_2(\Omega, H)$ satisfying the conditions $\sup_{t\in I} E\|\varphi(t)\|^2 < \infty$, where E is the expectation. An important subspace is given by $L_2^0(\Omega, H) = \{f \in L_2(\Omega, H) : f \text{ is } \mathfrak{F}_0 - \text{measurable}\}$.

We will also employ an axiomatic definition for the phase space $\mathfrak{B}$ which is similar to that used in [1]. Specifically, $\mathfrak{B}$ will be a linear space of functions mapping $(-\infty, 0]$ to H endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$ and verifying the following axioms:

If $x : (-\infty, \mu+\sigma] \to H, \sigma > 0$, is such that $x_\mu \in \mathfrak{B}$ and $x\Big|_{[\mu,\mu+\sigma]} \in \mathcal{C}([\mu, \mu+\sigma] : H)$, then for every $t \in [\mu, \mu+b]$ the following conditions hold:

(i) x_t is in $\mathfrak{B}$,

(ii) $\|x(t)\| \leq K_1\|x_t\|_{\mathfrak{B}}$,

(iii) $\|x_t\|_{\mathfrak{B}} \leq K_2(t-\mu)\sup\{\|x(s)\| : \mu \leq s \leq t\} + K_3(t-\mu)\|x_\mu\|_{\mathfrak{B}}$, where $K_1 > 0$ is a constant; $K_2, K_3 : [0, \infty) \to [1, \infty)$, K_2 is continuous, K_3 is locally bounded and K_1, K_2, K_3 are independent of $x(\cdot)$.

Reader may refer [1] for more examples of the phase spaces.

Suppose $x(t) : \Omega \to H_\alpha$, $t \leq a$, is a continuous $\mathfrak{F}_t$- adapted H_α-valued stochastic process. We can associate with another process $x_t : \Omega \to \mathcal{B}$, $t \geq 0$, setting

$$x_t = \{x(t+s)(w) : s \in (-\infty, 0]\}.$$

Throughout the paper $B_r(x, Z)$ represents the closed ball centered at x with radius $r > 0$ in Z and for a bounded function $\xi : J \to Z$ and $0 \leq t \leq a$. We use the notation $\|\xi\|_t$ for

$$\|\xi(\theta)\|_{Z,t} = \sup\{\|\xi(s)\|_Z : s \in [0, t]\}.$$

The collection of all strongly measurable, square-integrable H-valued random variables, denoted by $L_2(\Omega, \mathfrak{F}, P, H) \equiv L_2(\Omega, H)$ is a Banach space equipped with the norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot, w)\|_H^2)^{\frac{1}{2}}$, where the expectation E is defined by $E(h) = \int_\Omega h(w)dP$. For $J_1 = (-\infty, a], C(J_1, L_2(\Omega, H))$ denotes the Banach space of all continuous maps from J_1 into $L_2(\Omega, H))$ such that $\sup_{t\in J_1} E\|x(t)\|^2 < \infty$. An important subspace is given by $L_2^0(\Omega, H)) = \{f \in L_2(\Omega, H)) : f \text{ is } \mathfrak{F}_0 - \text{measurable}\}$.

Let Z be the closed subspace of all continuous processes x that belong to the space $C(J_1, L_2(\Omega, H))$ consisting of those $\mathfrak{F}_t$-adapted measurable processes for which the $\mathfrak{F}_0$-adapted process $\phi \in L_2(\Omega, \mathfrak{B})$. Let $\|\cdot\|_Z$ be a seminorm in Z defined by

$$\|x\|_Z = \sup_{t \in J} E(\|x_t\|_{\mathfrak{B}}^2)^{\frac{1}{2}}$$

where

$$E\|x_t\|_{\mathfrak{B}}^2 \leq \bar{K}_3 E\|\phi\|_{\mathfrak{B}}^2 + \bar{K}_2 \sup\{E\|x(s)\|^2 : 0 \leq s \leq \sigma\},$$

where $\bar{K}_3 = \sup_{t \in J}\{K_3(t)\}$ and $\bar{K}_2 = \sup_{t \in J}\{K_2(t)\}$. It is easy to verify that Z furnished with the norm topology as defined above is a Banach space.

Definition 2.1. *An $\mathfrak{F}_t$ adapted stochastic process $x(t) : J_1 \to H$ is a mild solution of the abstract Cauchy problem if the following hold:*

(i) *$x_0 = \phi \in \mathfrak{B}$ on J_0 satisfying $\|\phi\|_{\mathfrak{B}}^2 < \infty$;*

(ii) *The restriction of $x(\cdot)$ to the interval $[0, a)$ is a continuous stochastic process;*

(iii) *For each $s \in [0, t)$, the function $AT(t - s)F(s, x_s)$ is integrable, the following equation is satisfied a.e $t \in J$:*

$$\begin{aligned} x(t) &= T(t)[\phi(0) + F(0, \phi)] - F(t, x_t) - \int_0^t AT(t - s)F(s, x_s)ds + \int_0^t T(t - s)h(s, x_s)ds \\ &\quad + \int_0^t T(t - s)G(s, x_{\rho(s, x_s)})dW(s) \quad \text{for a.e. } \ t \in J, t > 0, \end{aligned} \tag{2}$$

Lemma 2.2 ([12]Leray-Schauder's fixed point Theorem). *Let D be a convex subset of a Banach space H and assume that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then, either the set $\{x \in D : x = \lambda F(x), \text{for some } 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

The proof of the main result of this paper relies heavily on the Leray-Schauder fixed point theorem.

3. MAIN RESULTS

In this section, the existence of mild solutions for the abstract Cauchy problem is to be studied. Throughout this section $\phi \in \mathfrak{B}$ is a fixed function, $(Y, \|\cdot\|_Y)$ is a Banach space continuously included in H. Throughout this section, the following two conditions are assumed to hold:

(H_Y) For every $y \in Y$, the function $t \to T(t)y$ is continuous from $[0, \infty)$ into Y. Moreover, $T(t)(Y) \subset D(A)$ for every $t > 0$ and there exists a positive function $\gamma \in L^1([0, a])$ such that $\|AT(t)\|_{L(Y,X)} \leq \gamma(t)$, for every $t \in J$.

(H_φ) For every $\Re(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in J \times \mathfrak{B}, \rho(s, \psi) \leq 0\}$. The function $t \to \varphi_t$ is continuous from $\Re(\rho^-)$ into $\mathfrak{B}$ and there exists a continuous and bounded function $\psi \in \mathfrak{B}, J^\varphi : \Re(\rho^-) \to (0, \infty)$ such that $E\|\varphi_t\|^2_{\mathfrak{B}} \leq J^\varphi(t)E\|\varphi\|^2_{\mathfrak{B}}$, for every $t \in \Re(\rho^-)$.

Let us first introduce the following hypotheses.

(H1) The function $G : J \times \mathfrak{B} \to H$ satisfies the following properties:

(i) For every $\psi \in \mathfrak{B}$, the function $t \to G(t, \psi)$ is strongly $\mathfrak{F}_t$-measurable.

(ii) For each $t \in J$ the function $G(t, \cdot) : \mathfrak{B} \to H$ is continuous.

(iii) There exists an integrable function $m : J \to [0, \infty)$ and a continuous non-decreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$E\|G(t, \psi)\|^2 \leq m(t)W(E\|\psi\|^2_{\mathfrak{B}}), \quad \text{for all } (t, \psi) \in J \times \mathfrak{B}.$$

(H2) The function F is Y-valued, $F : J \times \mathfrak{B} \to Y$ is continuous, and there exists positive constants c_1, c_2 such that

$$E\|F(t, \psi)\|^2_Y \leq c_1\|\psi\|^2_{\mathfrak{B}} + c_2, \text{ for all } (t, \psi) \in J \times \mathfrak{B}.$$

(H3) The function F is Y-valued, $F : J \times \mathfrak{B} \to Y$ is continuous, and there exists $L_F > 0$ such that

$$E\|F(t, \psi_1) - F(t, \psi_2)\|^2_Y \leq L_F^2 E\|\psi_1 - \psi_2\|^2_{\mathfrak{B}}, \text{ for all } (t, \psi_i) \in J \times \mathfrak{B}.$$

(H4) The function h is H-valued, $h : J \times \mathfrak{B} \to H$ is continuous, and there exists $L_h > 0$ such that

$$E\|h(t, \psi_1) - h(t, \psi_2)\|^2_Y \leq L_h^2 E\|\psi_1 - \psi_2\|^2_{\mathfrak{B}}, \text{ for all } (t, \psi_i) \in J \times \mathfrak{B}.$$

and there exists positive constants c_3, c_4 such that

$$E\|h(t, \psi)\|^2 \leq c_3\|\psi\|^2_{\mathfrak{B}} + c_4, \text{ for all } (t, \psi) \in J \times \mathfrak{B}.$$

(H5) Let $S(\varphi)$ be the space

$$S(\varphi) = \{x : J_1 \to H : x_0; x|_J \in C(J; L_2(\Omega, H))\}$$

endowed with the norm of the uniform convergence topology and $y : J_1 \to H$ be the function defined by $y_0 = \varphi$ on J_0 and $y(t) = T(t)\varphi(0)$ on J. Then, for every bounded set $\mathbb{Q}$ such that $\mathbb{Q} \subset S(\varphi)$, the set of functions $\{t \to F(t, x_t + y_t) : x \in \mathbb{Q}\}$ is equicontinuous on J.

Remark 3.1 Let $x(\cdot)$ be a function as in axiom (A). Let us mention that the conditions $(H_Y), (H2), (H3)$ are linked to the integrability of the function $s \to AT(t-s)F(s, x_s)$. In general, except for the trivial case in which A is a bounded linear operator, the operator function $t \to AT(t)$ is not integrable over J. However, if condition (H_Y) holds and F satisfies (H2) or (H3), then it follows from the Bochner's criterion for integrability and the estimate

$$\begin{aligned} E\|AT(t-s)F(s,x_s)\|^2 &= E\|AT(t-s)\|^2_{L(Y;H)}E\|F(s,x_s\|^2_Y \\ &\leq \gamma(t-s)\sup_{s\in J} E\|F(s,x_s\|^2_Y, \end{aligned}$$

that $s \to AT(t-s)F(s, x_s)$ is integrable over $[0, t)$, for every $t \in J$. For non-trivial examples of spaces Y for which condition (H_Y) is valid (see [16]).

The following Lemma can be easily verified using the phase space axioms. In the rest of this paper M_a and K_a are the constants defined by $M_a = \sup_{s\in J} M(s)$ and $K_a = \sup_{s\in J} K(s)$.

Lemma 3.2 Let $x : J_1 \to L_2(\Omega, H)$ such that $x_0 = \varphi$ and $x|_J \in C(J; H)$. Then

$$\|x_s\|_{\mathfrak{B}} \leq (M_a^2 + J_0^{\varphi 2})\|\varphi\|^2_{\mathfrak{B}} + K_a \sup \|x(\theta)\|_{\max\{0,s\}}, \quad s \in \Re(\rho^-) \cup J,$$

where $J_0^\varphi = \sup_{t\in\Re(\rho^-)} J^\varphi(t)$.

Theorem 3.3 Assume that conditions (H1), (H2)and (H3) are satisfied. If

$$8\left\{K_a^2\left[L_F^2\left(1+\int_0^a \gamma(s)ds\right) + \tilde{M}^2L_h^2 + \tilde{M}^2 Tr(Q)\liminf_{\xi\to\infty^+}\frac{W(\xi)}{\xi}\int_0^a m(s)ds+\right]\right\} < 1,$$

then there exists a mild solution of (1).

Proof : Consider the metric space $Y = \{u \in C(J; H) : u(0) = \varphi(0)\}$ endowed with the norm $\|u\|_a = \sup_{s\in J}\|u(s)\|$, and define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned} \Gamma x(t) &= T(t)(\varphi(0) + F(0,\varphi)) - F(t,\bar{x}_t) - \int_0^t AT(t-s)F(s,\bar{x}_s)ds + \int_0^t T(t-s)h(s,\bar{x}_s)ds \\ &\quad + \int_0^t T(t-s)G(s,\bar{x}_{\rho(s,\bar{x}_s)})dW(s), \quad s \in J, \end{aligned}$$

where $\bar{x} : J_1 \to H$ is defined by the relation $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J. From Axiom (A), the strong continuity of $(T(t))_{t\geq 0}$ and our assumptions on φ and G, we infer that $\Gamma x \in \mathcal{PC}$.

Let $\bar{\varphi} : J_1 \to H$ be the extension of φ to J_1 such that $\bar{\varphi}(\theta) = \varphi(0)$ on J. We affirm that there exists $r > 0$ such that $\Gamma(B_r(\bar{\varphi})|_J, Y)) \subset B_r(\bar{\varphi})|_J, Y)$. Indeed, if this property is false, then for every $r > 0$ there exists $x^r \in B_r(\bar{\varphi})|_J, Y)$ and $t^r \in J$ such that $r < E\|\Gamma x^r(t^r) - \varphi(0)\|^2$. Under these conditions, from Lemma 3.2 we find that

$$\begin{aligned}
r &< E\|\Gamma x^r(t^r) - \varphi(0)\|^2 \\
&\leq 8\Big\{E\|T(t^r)[\varphi(0) + F(0,\varphi)] - \varphi(0)\|^2 + E\|F(t^r, S(t^r)\varphi)\|^2 + E\|F(t^r, (\bar{x^r})_{t^r}) - F(t^r, S(t^r)\varphi)\|^2 \\
&\quad + \int_0^{t^r} E\|AT(t^r - s)\|^2_{L(Y;H)} E\|F(s, (\bar{x^r})_s) - F(s, S(s)\varphi)\|^2 ds \\
&\quad + \int_0^{t^r} E\|AT(t^r - s)\|^2_{L(Y;H)} E\|F(s, S(s)\varphi)\|^2 ds + \int_0^{t^r} E\|T(t^r - s)\|^2 E\|h(s, (\bar{x^r})_s) - h(s, S(s)\varphi)\|^2 ds \\
&\quad + \int_0^{t^r} E\|T(t^r - s)\|^2 E\|h(s, S(s)\varphi)\|^2 ds + Tr(Q)\int_0^{t^r} E\|G(s, \bar{x^r}_{\rho(s,\bar{x^r}_s)})\|^2_Q ds \\
&\leq 8\Big\{E\|T(t^r)[\varphi(0) + F(0,\varphi)] - \varphi(0)\|^2 + E\|F(s, S(s)\varphi)\|^2_a + L_F^2 K_a^2 E\|\bar{x^r} - \varphi(0)\|^2_{t^r} \\
&\quad + L_F^2 K_a^2 \int_0^{t^r} \gamma(t^r - s) E\|\bar{x^r}(s) - \varphi(0)\|^2_s ds + E\|F(s, S(s)\varphi)\|^2_a \int_0^{t^r} \gamma(s) ds + a\tilde{M}^2 L_h^2 K_a^2 E\|\bar{x^r} - \varphi(0)\|^2_s \\
&\quad + a\tilde{M}^2 E\|h(s, S(s)\varphi)\|^2_a + Tr(Q)\tilde{M}^2 \int_0^{t^r} m(s) W((M_a^2 + J_0^{\varphi 2})\|\varphi\|^2_{\mathfrak{B}} \\
&\quad + \tilde{M}^2 H^2 K_a^2 (E\|x^r(s) - \varphi(0)\|^2_s + E\|\varphi(0)\|^2))\Big\} \\
&\leq 8\Big\{E\|T(t^r)[\varphi(0) + F(0,\varphi)] - \varphi(0)\|^2 + E\|F(s, S(s)\varphi)\|^2_a + L_F^2 K_a^2 r + L_F^2 K_a^2 r \int_0^a \gamma(s) ds \\
&\quad + E\|F(s, S(s)\varphi)\|^2_a \int_0^a \gamma(s) ds + a\tilde{M}^2 L_h^2 K_a^2 r + a\tilde{M}^2 E\|h(s, S(s)\varphi)\|^2_a \\
&\quad + Tr(Q)\tilde{M}^2 \int_0^{t^r} m(s) W((M_a^2 + J_0^{\varphi 2})\|\varphi\|^2_{\mathfrak{B}} + \tilde{M}^2 H^2 K_a^2 (r + E\|\varphi(0)\|^2)) ds\Big\}
\end{aligned}$$

and hence

$$1 \leq 8\Big\{K_a^2 \Big(L_F^2 (1 + \int_0^a \gamma(s) ds) + \tilde{M}^2 L_h^2 + \tilde{M}^2 Tr(Q) \liminf_{\xi \to \infty^+} \frac{W(\xi)}{\xi} \int_0^a m(s) ds\Big)\Big\}$$

which is contrary to our assumption. Let $r > 0$ such that $\Gamma(B_r(\bar{\varphi})|_J, Y)) \subset B_r(\bar{\varphi})|_J, Y)$. In order to prove that $\Gamma(\cdot)$ is a continuous condensing map from $(B_r(\bar{\varphi})|_J, Y))$ into $(B_r(\bar{\varphi})|_J, Y))$, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\begin{aligned}\Gamma_1 x(t) &= T(t)(\varphi(0) + F(0,\varphi)) - F(t,\bar{x}_t) - \int_0^t AT(t-s)F(s,\bar{x}_s)ds + \int_0^t T(t-s)h(s,\bar{x}_s)ds \\ \Gamma_2 x(t) &= \int_0^t T(t-s)G(s,\bar{x}_{\rho(s,\bar{x}_s)})dW(s), \quad s \in J,\end{aligned}$$

From the proof of [9, Theorem 2.2], we know that Γ_2 is completely continuous. Moreover, from the phase space axioms and $(H4)$ we obtain

$$\|\Gamma_1 u(t) - \Gamma_1 v(t)\| \leq K_a^2(L_F^2 + \int_0^a \gamma(s)ds + \tilde{M}^2 L_h^2)E\|u-v\|_a^2 \text{ for a.e } t \in J$$

which proves that Γ_1 is a contraction on $(B_r(\bar{\varphi})|_J, Y))$ and so that Γ is a condensing operator on $B_r(\bar{\varphi})|_J, Y))$. Consequently, from the previous Remark and [12, Theroem 4.3.2] we deduce that the existence of a mild solution for the system (1)-(2). The proof is complete.

Theorem 3.4 Assume that $(H1)-(H2), (H4)-(H5)$ are satisfied. Further, assume that $\rho(t,\psi) \leq t$ for every $(t,\psi) \in J \times \mathfrak{B}$ and that $F: J \times \mathfrak{B} \to H$ is completely continuous. If

$$\mu = \Big[1 - 5\{c_1 K_a^2 + c_1 K_a^2 \int_0^a \gamma(s)ds + c_3 \tilde{M}^2 K_a^2\}\Big] > 0 \text{ and}$$
$$\frac{5\tilde{M}^2 K_a^2 Tr(Q)}{\mu}\int_0^a m(s)ds < \int_{\frac{D}{\mu}}^\infty \frac{ds}{W(s)},$$

where $D = (M_a^2 + J^{\varphi 2} + \tilde{M}^2 H^2 K_a^2)E\|\varphi\|_{\mathfrak{B}}^2 + \frac{K_a^2 C}{\mu}$ and $C = 5\Big\{\tilde{M}^2 E\|F(0,\varphi)\|^2 + E\|\varphi\|_{\mathfrak{B}}^2\Big[c_1 M_a^2 + c_1 M_a^2 + c_1 K_a^2 \tilde{M}^2 H^2 + c_1 M_a^2 \int_0^a \gamma s ds + c_1 \tilde{M}^2 K_a^2 H^2 \int_0^a \gamma(s)ds + c_3 a\tilde{M}^2 M_a^2 + c_3 a \tilde{M}^4 K_a^2 H^2\Big] + c_2(1 + \int_0^a \gamma(s)ds) + c_4 a\tilde{M}^2\Big\}$ then there exists a mild solution of (1)-(2).

Proof :On the space $\mathfrak{B}C = \{u : J_1 \to H; u_0 = 0, u|_J \in C(J,H)\}$ endowed with the norm $\|u\|_a = \sup_{s\in J}\|u(s)\|$, we define the operator $\Gamma: \mathfrak{B}C \to \mathfrak{B}C$ by

$$(\Gamma x)(t) = \begin{cases} 0, \text{ if } t \in J_0 \\ T(t)F(0,\varphi) - F(t,\bar{x}_t) - \int_0^t AT(t-s)F(s,\bar{x}_s)ds + \int_0^t T(t-s)h(s,\bar{x}_s)ds \\ + \int_0^t T(t-s)G(s,\bar{x}_{\rho(s,\bar{x}_s)})dW(s) \text{ for a.e.} \quad t \in J, \quad t > 0 \end{cases}$$

where $\bar{x} = x + y$ on $(-\infty, a]$. In order to use Lemma 2.2, we will establish a priori estimates for the solutions of the integral equation $z = \lambda\Gamma z, \lambda \in (0,1)$. Let x^λ be a solution of the integral equation $z = \lambda\Gamma z, \lambda \in (0,1)$ and $\alpha^\lambda(s) = \sup_{\theta\in J} E\|x^\lambda(\theta)\|^2$. If $t \in J$, from Lemma 3.2 and the fact that $\rho(s, \bar{x^\lambda_s} \leq s, s \in J$, we find that

$$\begin{aligned}
\mathbb{E}\|x^\lambda(t)\|^2 &\leq 5\Big\{\mathbb{E}\|T(t)F(0,\varphi)\|^2 + c_1\mathbb{E}\|(\bar{x^\lambda})_t\|^2_{\mathfrak{B}} + c_2 + \int_0^t \gamma(t-s)(c_1\mathbb{E}\|(\bar{x^\lambda})_s\|^2_{\mathfrak{B}} + c_2)ds \\
&\quad + \tilde{M}^2\int_0^t (c_3\mathbb{E}\|(\bar{x^\lambda})_t\|^2_{\mathfrak{B}} + c_4)ds + \tilde{M}^2 Tr(Q)\int_0^t m(s)W(\mathbb{E}\|(\bar{x^\lambda})_s\|)ds\Big|^2\Big\} \\
&\leq 5\Big\{\tilde{M}\mathbb{E}\|F(0,\varphi)\|^2 + c_1M_a^2\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + c_1K_a^2\tilde{M}^2H^2\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + c_2 + c_1K_a^2\mathbb{E}\|\alpha^\lambda\|^2_t \\
&\quad + c_1M_a^2\mathbb{E}\|\varphi\|^2_{\mathfrak{B}}\int_0^a \gamma(s)ds + c_1K_a^2\tilde{M}^2H^2\int_0^a\gamma(s)ds + c_2\int_0^a\gamma(s)ds \\
&\quad + c_1K_a^2(\alpha^\lambda(t))^2\int_0^a\gamma(s)ds + \tilde{M}^2\Big[ac_3M_a^2\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + ac_3K_a^2\tilde{M}^2H^2\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + ac_4 \\
&\quad + ac_3K_a^2(\alpha^\lambda(t))^2\Big] + \tilde{M}^2Tr(Q)\int_0^t m(s)W((M_a^2 + J^{\varphi 2} + K_a^2\tilde{M}^2H^2)\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} \\
&\quad + K_a^2(\alpha^\lambda(s))^2)ds\Big\} \\
&\leq 5\Big\{\tilde{M}\mathbb{E}\|F(0,\varphi)\|^2 + \mathbb{E}\|\varphi\|^2_{\mathfrak{B}}\Big[c_1M_a^2 + c_1K_a^2\tilde{M}^2H^2 + c_1M_a^2\int_0^a\gamma(s)ds \\
&\quad + c_1K_a^2\tilde{M}^2H^2\int_0^a\gamma(s)ds + c_3aM_a^2\tilde{M}^2 + c_3aK_a^2\tilde{M}^2H^2\Big] + c_2\Big(1 + \int_0^a\gamma(s)ds\Big) \\
&\quad + ac_4\tilde{M}^2 + \Big[c_1K_a^2 + c_1K_a^2\int_0^a\gamma(s)ds + c_3\tilde{M}^2K_a^2\Big](\alpha^\lambda(t))^2 \\
&\quad + \tilde{M}^2Tr(Q)\int_0^t m(s)W((M_a^2 + J^{\varphi 2} + K_a^2\tilde{M}^2H^2)\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + K_a^2(\alpha^\lambda(s))^2)ds\Big\}
\end{aligned}$$

Consequently,

$$(\alpha^\lambda(t))^2 \leq \frac{C}{\mu} + \frac{5\tilde{M}^2Tr(Q)}{\mu}\int_0^t m(s)W((M_a^2 + J^{\varphi 2} + K_a^2\tilde{M}^2H^2)\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + K_a^2(\alpha^\lambda(s))^2)ds$$

using the notation

$$\xi^\lambda(t) = (M_a^2 + J^{\varphi 2} + K_a^2\tilde{M}^2H^2)\mathbb{E}\|\varphi\|^2_{\mathfrak{B}} + K_a^2(\alpha^\lambda(t))^2$$

we obtain that

$$\xi^{\lambda}(t) \leq (M_a^2 + J^{\varphi 2} + K_a^2\tilde{M}^2H^2)\mathbb{E}\|\varphi\|_{\mathfrak{B}}^2 + \frac{K_a^2C}{\mu} + \frac{5\tilde{M}^2K_a^2Tr(Q)}{\mu}\int_0^t m(s)W(\xi^{\lambda}(s))ds$$

Denoting $\beta_\lambda(t)$ the right hand side of the above equation, it follow that

$$\beta_\lambda^{'}(t) \leq \frac{5\tilde{M}^2K_a^2Tr(Q)}{\mu}m(t)W(\beta_\lambda(t))$$

and hence

$$\int_{\beta_\lambda(0)=\frac{D}{\mu}}^{\beta_\lambda(t)} \frac{ds}{W(s)} \leq \frac{5\tilde{M}^2K_a^2Tr(Q)}{\mu}\int_0^a m(s)ds < \int_{\frac{D}{\mu}}^{\infty}\frac{ds}{W(s)}$$

which implies that the set of functions $\{\beta_\lambda(t) : \lambda \in (0,1)\}$ is bounded in $C(J;\mathfrak{B})$. Thus, the set of functions $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded on J.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ where $(\Gamma_i x)_0 = 0$ and

$$\Gamma_1 x(t) = T(t)F(0,\varphi) - F(t,x_t), \qquad t \in J$$

$$\Gamma_2 x(t) = -\int_0^t AT(t-s)F(s,x_s)ds, \qquad t \in J,$$

$$\Gamma_3 x(t) = \int_0^t T(t-s)h(s,x_s)ds + \int_0^t T(t-s)G(s,x_{\rho(s,x_s)})dW(s) \qquad t \in J.$$

The continuity of the function Γ_2 is easily shown. Moreover, from the proof of [8, Theorem 2.2] we know that Γ_3 is completely continuous. It remains to show that Γ_1 is completely continuous and that Γ_2 is a compact map. To this end, we first prove that Γ_1 is completely continuous. From the assumptions on G it follows that Γ_1 is a compact map. Let $(u^n)_{n\in N}$ be a sequence in $\mathcal{BC}$ and $u \in \mathcal{BC}$ such that $u^n \to u$. From the phase space axioms we infer that $u_s^n \to u_s$ uniformly on J as $n \to \infty$ and that the set $U \times \{u_s^n, u_s : s \in J, n \in N\}$ is relatively compact in $J \times \mathfrak{B}$. Thus, G is uniformly continuous on U, so that $F(s,u_s^n) \to F(s,u_s)$ uniformly on J as $n \to \infty$, which shows that Γ_1 is continuous and hence completely continuous.

Next, by using the Ascoli-Arzela theorem, we shall prove that Γ_2 is a compact map. In what follows, $B_r = B_r(0,\mathcal{BC})$

Step 1. The set $(\Gamma_2 B_r)(t) = \{\Gamma_2 x(t) : x \in B_r\}$ is relatively compact in H for each $t \in J$. The assertion clearly holds for $t = 0$. Let $0 < \epsilon < t \leq a$. For $u \in B_r$ we see that

$$\begin{aligned}\Gamma_2 u(t) &= T(\epsilon)\int_0^{t-\epsilon} AT(t-\epsilon-s)F(s,u_s)ds + \int_{t-\epsilon}^{t} AT(t-s)F(s,u_s)ds \\ &\in T(\epsilon)\Big\{x \in H : \|x\| \leq (c_1 K_a^2 r + c_2)\int_0^a \gamma(s)ds\Big\} + B_{r^*}(0,H)\end{aligned}$$

where $r^* = (c_1 K_a^2 r + c_2)\int_{t-\epsilon}^{t} \gamma(s)ds$. From this we can infer that $(\Gamma_2 B_r)(t)$ is totally bounded in H and hence relatively compact in H.

Step 2. The set of functions $\Gamma_2 B_r = \Big\{\Gamma_2 x(t) : x \in B_r\Big\}$ is equicontinuous on J.

Let $t \in (0,a)$. For $u \in B_r$ and $h > 0$ such that $t + h \in [0,a]$, we obtain

$$\begin{aligned}\mathbb{E}\|\Gamma_2 u(t+h) - \Gamma_2 u(t)\|^2 &= \mathbb{E}\Big\|(T(h)-I)\Gamma_2 u(t) + \int_t^{t+h} AT(t-s)F(s,u_s)ds\Big\|^2 \\ &\leq \mathbb{E}\|(T(h)-I)\Gamma_2 u(t)\| + (c_1 K_a^2 r + c_2)\int_t^{t+h} \gamma(s)ds.\end{aligned}$$

Since the set $(\Gamma_2 B_r)(t)$ is relatively compact in H and $(T(t))_{t\geq 0}$ is strongly continuous, it follows that $\|(T(h)-I)\Gamma_2 u(t)\| \to 0$ uniformly for $u \in B_r$, which from the last inequality enables us to conclude that $\Gamma_2 B_r$ is right equicontinuous at $t \in (0,a)$. In a similar manner we can prove that $\Gamma_2 B_r$ is left continuous at zero and left continuous at $t \in (0,a]$. This completes the proof that Γ_2 is completely continuous.

These remarks, in conjunctions with Lemma 2.2, show that Γ has a fixed point $u \in \mathcal{BC}$. Clearly, the function $x = u + y$ is a mild solution of (1).

4. **CONCLUSION**

The existence of mild solutions for a stochastic neutral differential equations has been investigated in this paper. Based on Leray-Schauder fixed point theorem, the sufficient conditions of the existence of solution for the system with state dependent delay has been obtained.

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DEVELOPMENT OF INVENTORY MODEL OF DETERIORATING ITEMS WITH LIFE TIME UNDER TRADE CREDIT AND TIME DISCOUNTING

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ABSTRACT :
In this paper, an inventory model has been developed for deteriorating items with life time under the assumptions of trade credit and time discounting. Time horizon has been considered finite. The demand rate has been taken linear function of time and deterioration rate has been taken constant. The time horizon has been divided into n equal sub-intervals.
Keywords: ***Deterioration, life time, trade credit and discount.***

1 INTRODUCTION:

In the classical EOQ model, it is assumed that the retailer must be paid for the items at the time of delivery. However, in real life situation, the supplier may offer the retailer a delay period, which is called the trade credit period, for the payment of purchasing cost to stimulate his products. During the trade credit period, the retailer can sell the products and can earn the interest on the revenue thus obtained. It is beneficial for the retailer to delay the settlement of the replenishment account up to the last moment of the permissible delay allowed by the supplier.

Several researchers discussed the inventory problems under the permissible delay in payment condition. **Goyal** (1985) discussed a single item inventory model under permissible delay in payment. **Chung** (1998) used an alternative approach to obtain the economic order quantity under permissible delay in payment. **Agrawal & Jaggi** (1995) discussed the inventory model with an exponential deterioration rate under permissible delay in payment. **Chang et. al.** (2002) extended this work for variable deterioration rate. **Liao, et al.** (2000) discussed the topic with inflation. **Jamal, et. al**. (1997) and **Chang & Dye** (2001) extended this work with shortages. **Chen & Chung** (1999) discussed light buyer's economic order model under trade credit. **Huang & Shinn** (1997) studied an inventory system for retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payment. **Jamal, et. al.** (1997) and **Sarkar, et. al.** (2000) obtained the optimal time of payment under permissible delay in payment. **Teng** (2002) considered the selling price not equal to the purchasing price to modify the model under permissible delay in payment. **Shinn & Huang** (2003) obtained the optimal price and order size simultaneously under the condition of order-size dependent delay in payments. They assumed the length of the credit period as a function of retailer's order size and the demand rate to be the function of selling price. **Chung & Huang** (2003) extended this work within the EPQ framework and obtained the retailer's optimal ordering policy. **Huang** (2003) extended this work under two level of trade credit. **Huang** (2005) modify the **Goyal's** model under the following assumptions:

(i) The unit selling price and the unit purchasing price are not necessarily equal.

(ii) The supplier offers the retailer partial trade credit, i.e. the retailer has to make a partial payment to the supplier in the beginning and has to pay the remaining balance at the end of the permissible delay period.

He adopted the cost minimization technique to investigate the optimal retailer's inventory policy. **Megha Rani, Hari Kishan and Shiv Raj Singh** (2011) discussed the inventory model of deteriorating products under supplier's partial trade credit policy.

While determining the optimal ordering policy, the effect of inflation and time value of money cannot be ignored. **Buzacott** (1975) developed an EOQ model with inflation subject to different types of pricing policies. **Hou** (2006) developed an inventory model with deterioration, inflation, time value of money and stock dependent consumption rate. Recently **Madhu & Deepa** (2010) **Harikishan etal** (2012 & 2015) developed inventory models of deteriorating product with lifetime & variable deterioration rate with declining demand.

In most of the inventory models it is assumed that the deterioration of items starts in the very beginning of the inventory. In real life problems the deterioration of items starts after some time which is known as life time. In this paper, an inventory model has been developed for deteriorating items with life time under the assumptions of life time, trade credit and time discounting.

2. ASSUMPTIONS AND NOTATIONS:

Assumptions:

The following assumptions are considered in this paper:

(i) The demand rate is linear function of time which is given by $at+b$.

(ii) Time horizon is finite given by H.

(iii) Shortages are not allowed.

(iv) The deteriorating rate is deterministic and constant. Deterioration starts after time μ which is the life time.

(vi) During the time the account is not settled, the generated sales revenue is deposited in an interest bearing account. When $T \geq M$, the account is settled at $T=M$ and we start paying for the interest charges on items in stock. When $T \leq M$, the account is settled at $T=M$ and we need not to pay any interest charge.

Notations:

The following notations have been used in this chapter:

(i) The demand rate=$at+b$ where a and b are positive constants.

(ii) A = the ordering cost per order.

(iii) c= the unit purchasing price.

(iv) h=unit holding cost per unit time excluding interest charges.

(v) M= the trade credit period.

(vi) H= length of planning horizon.

(vii) T=the replenishment cycle time in years.

(viii) n= number of replenishment during the planning horizon.

(ix) θ =the deterioration rate of the on hand inventory.

(x) μ = the life time.

(xi) I_e = interest earned per Re per year.

(xii) I_c = interest charged per Re per year.

(xiii) $I(t)$ = stock level at any time t.

(xiv) Q= maximum stock level.

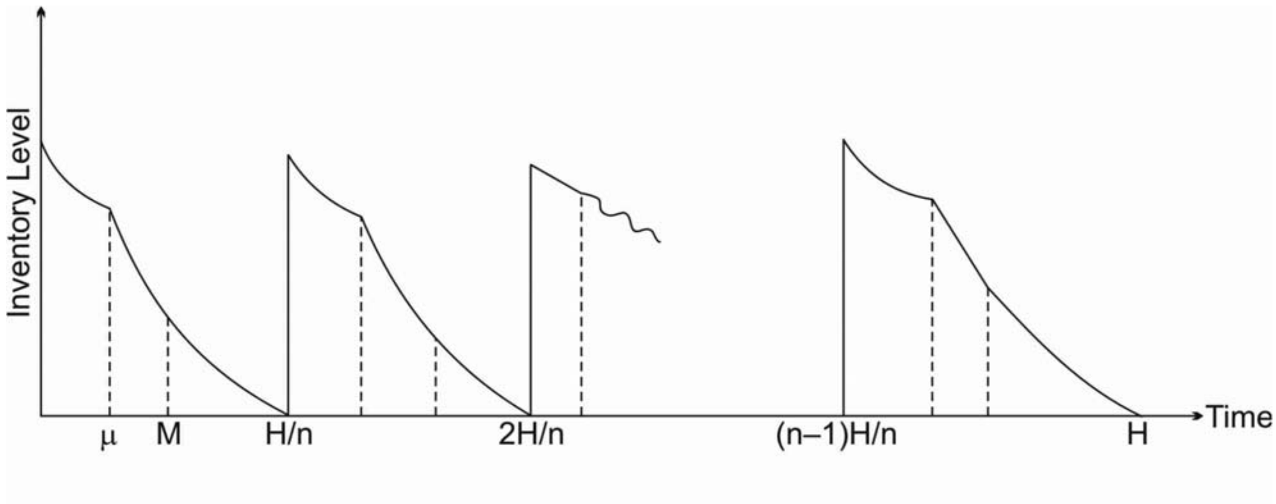
(xv) $TVC(T)$= the annual total relevant cost, which is a function of T.

(xvi) T^* = the optimal cycle time of $TVC(T)$.

(xvii) Q^* = the optimal order quantity.

3. MATHEMATICAL MODEL:

The total time horizon H has been divided in n equal parts of length T so that $T = \frac{H}{n}$. Therefore the reorder times over the planning horizon H are given by $T_j = jT$, $(j = 0,1,2...,n-1)$. This model is given by fig. 1.



(Fig. 1)

Let $I(t)$ be the inventory level during the first replenishment cycle. This inventory level is depleted due to demand during life time and due to demand and deterioration after life time. The governing differential equations of the stock status during the period $0 \le t \le T$ are given by

$$\frac{dI}{dt} = -(at+b), \qquad 0 \le t \le \mu \qquad \text{...(1)}$$

$$\frac{dI}{dt} + \theta I = -(at+b). \qquad \mu \le t \le T \qquad \text{...(2)}$$

The boundary conditions are

$$I(0) = Q, \qquad \text{...(3)}$$

$$I(T) = 0. \qquad \text{...(4)}$$

4. Analysis:

Solving equation (1) and using boundary condition (3), we get

$$I(t) = Q - \left(a\frac{t^2}{2} + bt\right). \qquad \text{...(5)}$$

Solving equation (2) and using boundary condition (4), we get

$$I(t) = -\frac{1}{\theta}(at+b) + \frac{a}{\theta^2} + \left[\frac{1}{\theta}(aT+b) - \frac{a}{\theta^2}\right]e^{\theta(T-t)}. \qquad \text{...(6)}$$

The continuity of $I(t)$ at $t = \mu$ gives

$$I(\mu) = Q - \left(a\frac{\mu^2}{2} + b\mu\right)$$

$$= -\frac{1}{\theta}(a\mu+b) + \frac{a}{\theta^2} + \left[\frac{1}{\theta}(aT+b) - \frac{a}{\theta^2}\right]e^{\theta(T-\mu)}.$$

This provides

$$I(0)=Q=\left(a\frac{\mu^2}{2}+b\mu\right)-\frac{1}{\theta}(a\mu+b)+\frac{a}{\theta^2}+\left[\frac{1}{\theta}(aT+b)-\frac{a}{\theta^2}\right]e^{\theta(T-\mu)}. \quad \ldots(7)$$

The present value of the total replenishment costs is given by

$$C_R=A\sum_{j=0}^{n-1}e^{-jRT}$$

$$=A\frac{\left(1-e^{-RH}\right)}{\left(1-e^{-\frac{RH}{n}}\right)}. \quad \ldots(8)$$

The present value of the total purchasing costs is given by

$$C_p=c\sum_{j=0}^{n-1}I(0)e^{-jRT}$$

$$=c\left[\left(a\frac{\mu^2}{2}+b\mu\right)-\frac{1}{\theta}(a\mu+b)+\frac{a}{\theta^2}+\left[\frac{1}{\theta}(aT+b)-\frac{a}{\theta^2}\right]e^{\theta(T-\mu)}\right]\frac{\left(1-e^{-RH}\right)}{\left(1-e^{-\frac{RH}{n}}\right)}. \quad \ldots(9)$$

The present value of the holding cost is given by

$$h_1=h\left[\int_0^{\mu}I(t)dt+\int_{\mu}^{T}I(t)dt\right]$$

$$=h\left[\int_0^{\mu}\left\{Q-\left(a\frac{t^2}{2}+bt\right)\right\}e^{-Rt}dt\right.$$

$$\left.+\int_{\mu}^{T}\left\{-\frac{1}{\theta}(at+b)+\frac{a}{\theta^2}+\left[\frac{1}{\theta}(aT+b)-\frac{a}{\theta^2}\right]e^{\theta(T-t)}\right\}e^{-Rt}dt\right]$$

$$=h\left[\frac{Q}{R}\left(1-e^{-R\mu}\right)+\frac{a}{2}\left(\frac{\mu^2}{R}e^{-R\mu}+\frac{2\mu}{R^2}e^{-R\mu}+\frac{2}{R^3}e^{-R\mu}-\frac{2}{R^3}\right)\right.$$

$$+b\left(\frac{\mu}{R}e^{-R\mu}+\frac{1}{R^2}e^{-R\mu}-\frac{1}{R^2}\right)+\frac{a}{\theta}\left(\frac{T}{R}e^{-RT}+\frac{1}{R^2}e^{-RT}-\frac{\mu}{R}e^{-R\mu}-\frac{1}{R^2}e^{-R\mu}\right)$$

$$\left.+\frac{b}{\theta}\left(\frac{e^{-RT}}{R}-\frac{e^{-R\mu}}{R}\right)-\frac{a}{\theta^2}\left(\frac{e^{-RT}}{R}-\frac{e^{-R\mu}}{R}\right)+\left[\frac{\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}}{\theta+R}\right]\left[e^{\theta T-(\theta+R)\mu}-\frac{e^{-RT}}{\theta+R}\right]\right]. \quad \ldots(10)$$

The present value of the total holding cost is given by

$$C_h=\sum_{j=0}^{n-1}h_1e^{-jRT}$$

$$= h\left[\frac{Q}{R}\left(1-e^{-R\mu}\right)+\frac{a}{2}\left(\frac{\mu^2}{R}e^{-R\mu}+\frac{2\mu}{R^2}e^{-R\mu}+\frac{2}{R^3}e^{-R\mu}-\frac{2}{R^3}\right)\right.$$

$$+b\left(\frac{\mu}{R}e^{-R\mu}+\frac{1}{R^2}e^{-R\mu}-\frac{1}{R^2}\right)+\frac{a}{\theta}\left(\frac{T}{R}e^{-RT}+\frac{1}{R^2}e^{-RT}-\frac{\mu}{R}e^{-R\mu}-\frac{1}{R^2}e^{-R\mu}\right)$$

$$\left.+\frac{b}{\theta}\left(\frac{e^{-RT}}{R}-\frac{e^{-R\mu}}{R}\right)-\frac{a}{\theta^2}\left(\frac{e^{-RT}}{R}-\frac{e^{-R\mu}}{R}\right)+\left[\frac{\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}}{\theta+R}\right]\left[e^{\theta T-(\theta+R)\mu}-\frac{e^{-RT}}{\theta+R}\right]\right]$$

$$\times\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right). \qquad \ldots(11)$$

Case 1: $M \le T = \dfrac{H}{n}$**.**

Sub case I: $0 \prec M \le \mu$**.**

In this case, the interest payable is given by

$$I_{p1}^{1} = cI_c\left[\int_M^{\mu} I(t)e^{-Rt}dt+\int_{\mu}^{T} I(t)e^{-Rt}dt\right]$$

$$= cI_c\left[-\frac{Qe^{-R\mu}}{R}+\frac{Qe^{-RT}}{R}+\frac{a}{2}\left[\frac{\mu^2e^{-R\mu}}{R}+\frac{2\mu e^{-R\mu}}{R^2}+\frac{2e^{-R\mu}}{R^3}-\frac{M^2e^{-RM}}{R}\right.\right.$$

$$\left.-\frac{2Me^{-RM}}{R^2}-\frac{2e^{-RM}}{R^3}\right]+b\left[\frac{\mu e^{-R\mu}}{R}+\frac{e^{-R\mu}}{R^2}-\frac{Me^{-RM}}{R}-\frac{e^{-RM}}{R^2}\right]$$

$$+\frac{a}{\theta}\left(\frac{Te^{-RT}}{R}+\frac{e^{-RT}}{R^2}\right)+\frac{b}{\theta}\frac{e^{-RT}}{R^2}-\frac{a}{\theta}\frac{e^{-RT}}{R^2}-\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{-RT}}{(\theta+r)}$$

$$\left.-\frac{a}{\theta}\left(\frac{\mu e^{-R\mu}}{R}+\frac{e^{-R\mu}}{R^2}\right)-\frac{b}{\theta}\frac{e^{-R\mu}}{R^2}+\frac{a}{\theta}\frac{e^{-R\mu}}{R^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-R\mu}}{(\theta+r)}\right]. \qquad \ldots(12)$$

The present value of the total interest payable over the time horizon *H* is given by

$$I_{p1}^{H} = \sum_{j=0}^{n-1} I_{p1}^{1}e^{-jRT}$$

$$= \left[cI_c\left[-\frac{Qe^{-R\mu}}{R}+\frac{Qe^{-RT}}{R}+\frac{a}{2}\left[\frac{\mu^2e^{-R\mu}}{R}+\frac{2\mu e^{-R\mu}}{R^2}+\frac{2e^{-R\mu}}{R^3}-\frac{M^2e^{-RM}}{R}\right.\right.\right.$$

$$\left.-\frac{2Me^{-RM}}{R^2}-\frac{2e^{-RM}}{R^3}\right]+b\left[\frac{\mu e^{-R\mu}}{R}+\frac{e^{-R\mu}}{R^2}-\frac{Me^{-RM}}{R}-\frac{e^{-RM}}{R^2}\right]$$

$$+\frac{a}{\theta}\left(\frac{Te^{-RT}}{R}+\frac{e^{-RT}}{R^2}\right)+\frac{b}{\theta}\frac{e^{-RT}}{R^2}-\frac{a}{\theta}\frac{e^{-RT}}{R^2}-\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{-RT}}{(\theta+r)}$$

$$-\frac{a}{\theta}\left(\frac{\mu e^{-R\mu}}{R}+\frac{e^{-R\mu}}{R^2}\right)-\frac{b}{\theta}\frac{e^{-R\mu}}{R^2}+\frac{a}{\theta}\frac{e^{-R\mu}}{R^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-R\mu}}{(\theta+r)}\Bigg]$$

$$+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-R\mu}}{(\theta+r)}\Bigg]\Bigg]\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right). \qquad \ldots(13)$$

The present value of the total interest earned during the first replenishment cycle is given by

$$I_{e1}^{1}=cI_e\int_0^T(at+b)te^{-Rt}dt$$

$$=cI_e\left[a\left(-\frac{T^2e^{-RT}}{R}-\frac{2Te^{-RT}}{R^2}-\frac{2e^{-RT}}{R^3}+\frac{2}{R^3}\right)\right.$$

$$\left.+b\left(\frac{Te^{-RT}}{R}-\frac{e^{-RT}}{R^2}+\frac{2}{R^2}\right)\right]. \qquad \ldots(14)$$

Hence the present value of the total interest earned over the time horizon H is given by

$$I_{e1}^{H}=\sum_{j=0}^{n-1}I_{e1}^{1}e^{-jRT}$$

$$=cI_e\left[a\left(-\frac{T^2e^{-RT}}{R}-\frac{2Te^{-RT}}{R^2}-\frac{2e^{-RT}}{R^3}+\frac{2}{R^3}\right)\right.$$

$$\left.+b\left(\frac{Te^{-RT}}{R}-\frac{e^{-RT}}{R^2}+\frac{2}{R^2}\right)\right]\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right).$$

Sub case II: $\mu > M$.

In this case, the interest payable is given by

$$I_{p1}^{2}=cI_c\left[\int_M^T I(t)e^{-Rt}dt\right]$$

$$=cI_c\int_M^T\left[-\frac{1}{\theta}(at+b)+\frac{a}{\theta^2}+\left[\frac{1}{\theta}(aT+b)-\frac{a}{\theta^2}\right]e^{\theta(T-t)}\right]e^{-RT}dt$$

$$=cI_c\left[\frac{a}{\theta}\left(\frac{Te^{-RT}}{R}+\frac{e^{-RT}}{R^2}\right)+\frac{b}{\theta}\frac{e^{-RT}}{R}-\frac{ae^{-RT}}{\theta^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{-RT}}{\theta+R}\right.$$

$$\left.-\frac{a}{\theta}\left(\frac{Me^{-RM}}{R}+\frac{e^{-RM}}{R^2}\right)-\frac{b}{\theta}\frac{e^{-RM}}{R}-\frac{ae^{-RM}}{\theta^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-RM}}{\theta+R}\right].$$

$$\ldots(15)$$

The present value of the total interest payable over the time horizon H is given by

$$I_{p1}^{H}=\sum_{j=0}^{n-1}I_{p1}^{2}e^{-jRT}$$

$$=cI_c\left[\frac{a}{\theta}\left(\frac{Te^{-RT}}{R}+\frac{e^{-RT}}{R^2}\right)+\frac{b}{\theta}\frac{e^{-RT}}{R}-\frac{ae^{-RT}}{\theta^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{-RT}}{\theta+R}\right.$$

$$-\frac{a}{\theta}\left(\frac{Me^{-RM}}{R}+\frac{e^{-RM}}{R^2}\right)-\frac{b}{\theta}\frac{e^{-RM}}{R}-\frac{ae^{-RM}}{\theta^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-RM}}{\theta+R}\Bigg]$$

$$\times\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right). \qquad \ldots(16)$$

Therefore the total present value of the costs over the time horizon H is given by

$$TVC_1(n)=C_R+C_p+C_h+I_{p1}^H-I_{e1}^H$$

$$=\Bigg[A+c\Bigg[\left(a\frac{\mu^2}{2}+b\mu\right)-\frac{1}{\theta}(a\mu+b)+\frac{a}{\theta^2}+\left[\frac{1}{\theta}(aT+b)-\frac{a}{\theta^2}\right]e^{\theta(T-\mu)}\Bigg]$$

$$+h\Bigg[\frac{Q}{R}\left(1-e^{-R\mu}\right)+\frac{a}{2}\left(\frac{\mu^2}{R}e^{-R\mu}+\frac{2\mu}{R^2}e^{-R\mu}+\frac{2}{R^3}e^{-R\mu}-\frac{2}{R^3}\right)$$

$$+b\left(\frac{\mu}{R}e^{-R\mu}+\frac{1}{R^2}e^{-R\mu}-\frac{1}{R^2}\right)+\frac{a}{\theta}\left(\frac{T}{R}e^{-RT}+\frac{1}{R^2}e^{-RT}-\frac{\mu}{R}e^{-R\mu}-\frac{1}{R^2}e^{-R\mu}\right)$$

$$+\frac{b}{\theta}\left(\frac{e^{-RT}}{R}-\frac{e^{-R\mu}}{R}\right)-\frac{a}{\theta^2}\left(\frac{e^{-RT}}{R}-\frac{e^{-R\mu}}{R}\right)+\left[\frac{\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}}{\theta+R}\right]\left[e^{\theta T-(\theta+R)\mu}-\frac{e^{-RT}}{\theta+R}\right]\Bigg]$$

$$+\Bigg[cI_c\Bigg[-\frac{Qe^{-R\mu}}{R}+\frac{Qe^{-RT}}{R}+\frac{a}{2}\Bigg[\frac{\mu^2e^{-R\mu}}{R}+\frac{2\mu e^{-R\mu}}{R^2}+\frac{2e^{-R\mu}}{R^3}-\frac{M^2e^{-RM}}{R}$$

$$-\frac{2Me^{-RM}}{R^2}-\frac{2e^{-RM}}{R^3}\Bigg]+b\left[\frac{\mu e^{-R\mu}}{R}+\frac{e^{-R\mu}}{R^2}-\frac{Me^{-RM}}{R}-\frac{e^{-RM}}{R^2}\right]$$

$$+\frac{a}{\theta}\left(\frac{Te^{-RT}}{R}+\frac{e^{-RT}}{R^2}\right)+\frac{b}{\theta}\frac{e^{-RT}}{R^2}-\frac{a}{\theta}\frac{e^{-RT}}{R^2}-\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{-RT}}{(\theta+r)}$$

$$-\frac{a}{\theta}\left(\frac{\mu e^{-R\mu}}{R}+\frac{e^{-R\mu}}{R^2}\right)-\frac{b}{\theta}\frac{e^{-R\mu}}{R^2}+\frac{a}{\theta}\frac{e^{-R\mu}}{R^2}+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-R\mu}}{(\theta+r)}\Bigg]$$

$$+\left(\frac{(aT+b)}{\theta}-\frac{a}{\theta^2}\right)\frac{e^{\theta(T-\mu)-R\mu}}{(\theta+r)}\Bigg]\Bigg]-cI_e\Bigg[a\left(-\frac{T^2e^{-RT}}{R}-\frac{2Te^{-RT}}{R^2}-\frac{2e^{-RT}}{R^3}+\frac{2}{R^3}\right)$$

$$+b\left(\frac{Te^{-RT}}{R}-\frac{e^{-RT}}{R^2}+\frac{2}{R^2}\right)\Bigg]\Bigg]\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right). \qquad \ldots(17)$$

Case 2: $M > T = \dfrac{H}{n}$**:**

In this case, the interest charged by the supplier will be zero because the supplier can be paid at *M* which is greater than *T*. The interest earned in the first cycle is the interest earned during the period (0,*T*) and the interest earned from the cash invested during the period (*T,M*). This is given by

$$I_{e2}^1=cI_e\left[\int_0^T(at+b)te^{-Rt}dt+(M-T)e^{-RT}\int_0^T(at+b)dt\right]$$

$$= cI_e\left[-\frac{(aT^2+bT)}{R}e^{-RT} - \frac{(2aT+b)}{R^2}e^{-RT} + \frac{b}{R^2} - \frac{2a}{R^3}e^{-RT} + \frac{2a}{R^3}\right.$$
$$\left.+(M-T)e^{-RT}\left(\frac{aT^2}{2}+bT\right)\right]. \qquad \ldots(18)$$

Hence the present value of the total interest earned over the time horizon H is given by

$$I_{e2}^{H} = \sum_{j=0}^{n-1} I_{e2}^{1} e^{-jRT}$$
$$= cI_e\left[-\frac{(aT^2+bT)}{R}e^{-RT} - \frac{(2aT+b)}{R^2}e^{-RT} + \frac{b}{R^2} - \frac{2a}{R^3}e^{-RT} + \frac{2a}{R^3}\right.$$
$$\left.+(M-T)e^{-RT}\left(\frac{aT^2}{2}+bT\right)\right]\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right). \qquad \ldots(19)$$

As the replenishment cost, purchasing cost and holding cost over the time horizon H are the same as in case I, the total present value of the costs is given by

$$TVC_2(n) = C_R + C_p + C_h - I_{e2}^{H}$$
$$= [A + c\left[\left(a\frac{\mu^2}{2}+b\mu\right) - \frac{1}{\theta}(a\mu+b) + \frac{a}{\theta^2} + \left[\frac{1}{\theta}(aT+b) - \frac{a}{\theta^2}\right]e^{\theta(T-\mu)}\right]$$
$$+ h\left[\frac{Q}{R}\left(1-e^{-R\mu}\right) + \frac{a}{2}\left(\frac{\mu^2}{R}e^{-R\mu} + \frac{2\mu}{R^2}e^{-R\mu} + \frac{2}{R^3}e^{-R\mu} - \frac{2}{R^3}\right)\right.$$
$$+ b\left(\frac{\mu}{R}e^{-R\mu} + \frac{1}{R^2}e^{-R\mu} - \frac{1}{R^2}\right) + \frac{a}{\theta}\left(\frac{T}{R}e^{-RT} + \frac{1}{R^2}e^{-RT} - \frac{\mu}{R}e^{-R\mu} - \frac{1}{R^2}e^{-R\mu}\right)$$
$$\left.+ \frac{b}{\theta}\left(\frac{e^{-RT}}{R} - \frac{e^{-R\mu}}{R}\right) - \frac{a}{\theta^2}\left(\frac{e^{-RT}}{R} - \frac{e^{-R\mu}}{R}\right) + \left[\frac{\frac{(aT+b)}{\theta} - \frac{a}{\theta^2}}{\theta+R}\right]\left[e^{\theta T-(\theta+R)\mu} - \frac{e^{-RT}}{\theta+R}\right]\right]$$
$$+ cI_e\left[\frac{(aT^2+bT)}{R}e^{-RT} + \frac{(2aT+b)}{R^2}e^{-RT} - \frac{b}{R^2} + \frac{2a}{R^3}e^{-RT} - \frac{2a}{R^3}\right.$$
$$\left.+(M-T)e^{-RT}\left(\frac{aT^2}{2}+bT\right)\right]\left(\frac{1-e^{-RH}}{1-e^{-RH/n}}\right). \qquad \ldots(20)$$

At $M=T=\frac{H}{n}$, we have $TVC_1(n)=TVC_2(n)$. Thus we have

$$TVC(n) = \begin{cases} TVC_1(n), T = \frac{H}{n} \geq M \\ TVC_2(n), T = \frac{H}{n} \leq M \end{cases}$$

5. Algorithm: The following algorithm is used to derive the optimal values of n, T, Q and $TVC(n)$:

Step 1. Start by choosing a discrete value of n equal or greater than 1.

Step 2. If $T=\frac{H}{n}\geq M$ for different integral values of n then $TVC_1(n)$ is derived from expression (6.17). If $T=\frac{H}{n}\leq M$ for different integral values of *n* then $TVC_2(n)$ is derived from expression (6.20).

Step 3. Step 1 and 2 are repeated for all possible values of *n* with $T=\frac{H}{n}\geq M$ until the minimum value of $TVC_1(n)$ is found from expression (6.17). Let $n=n_1^*$ be such value of *n*. For all possible values of n with $T=\frac{H}{n}\leq M$ until the minimum value of $TVC_2(n)$ is found from expression (6.20). Let $n=n_2^*$ be such value of *n*. The values $n_1^*, n_2^*, TVC_1(n^*)$ and $TVC_2(n^*)$ constitute the optimal solution.

Step 4. The optimal number of replenishment *n** is selected such that

$$TVC(n^*)=\begin{cases} TVC_1(n), T=\frac{H}{n^*}\geq M \\ TVC_2(n), T=\frac{H}{n^*}\leq M \end{cases}$$

The optimal value of ordered quantity *Q** is derived by substituting *n** in the expression (7) and optimal cycle time *T** is given by $T=\frac{H}{n^*}$.

6. NUMERICAL EXAMPLE:

In the demand rate *a*=1, *b*=1000, replenishment cost *A*=$ 800 per order, the holding cost *h*=$ 2 per unit per year, per unit item cost *c*=$ 8 per unit, rate of deterioration $\theta=0.15$, net discount rate of inflation *R*=$ 0.1 per $ per year, the interest earned per $ per year by supplier $I_e=0.15/\$$, the interest charged per $ per year by supplier $I_e=0.12/\$$, planning horizon *H*=5 years, $\mu=50$ days(=0.139 years) and *M*=60 days(=0.167 years) by assuming 360 days per year.

By the application of computational procedure, the results are shown in the following table:

	No. of replenishment n	Cycle time T	Order quantity Q	Total present value of cost.
Case I	26	0.192	147.75	28322.24
	27	0.185	140.57	28321.37
	28	0.179	134.57	28323.12
Case II	37	0.135	90.56	28276.26
	38	0.132	87.57*	28275.59*
	39	0.128	83.56	28277.11

*quantities are the optimal quantities.

7. CONCLUSIONS:

In this paper, an inventory model has been developed for deteriorating items with life time under the assumptions of trade credit and time discounting. Time horizon has been considered finite. The demand rate has been taken linear function of time and deterioration rate has been taken constant. The time horizon has been divided into *n* equal sub-intervals. This work can further be extended for other forms of demand rate, for variable deterioration rate and for multi items.

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CASUAL MODEL OF ACADEMIC PERFORMANCE OF STUDENTS OF SECONDARY SCHOOLS-A STRUCTURAL EQUATION MODELING APPROACH

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ABSTRACT :

Aim: The purpose of this study is to examine the contributory relationships among factors of teachers of secondary schools influential in academic performance of students by exploiting the applications of Structural Equation Modeling (SEM). A simple random sample of 30 schools with 60 teachers and 300 students was chosen and data were collected on four scales like study habits of students, organizational climate, teaching effectiveness and job satisfaction of teachers of secondary schools through direct personal interview method. The casual relationships were established by structural equation modeling method using SPSS and AMOS statistical software. The organizational climate and job satisfaction of teachers have significant positive relationship with academic performance of students ($p<0.05$). The SEM was fitted to the academic performance of students data adequately. The results indicated that, the organizational climate and job satisfaction of teachers had significant effect on academic performance of students ($p<0.05$). The organizational climate and job satisfaction of teachers are the main contributors to academic performance of students of secondary schools in Shigoan, Karnataka State, India. It is proposed to increase the awareness of importance of academic performance of students.

Keywords: ***Structural Equation Model, SEM, Modeling, Academic performance***

1. INTRODUCTION

Education is an essential human virtue. Education is bringing out the best already in human. Educational institutions have always been under the critical microscope of an ever-demanding society. As stakeholders clamor for quality education, school leaders often exert efforts into responsive mechanisms that could lead to school improvement and increased student performance. There is a growing body of literature from researchers and educationists which have made an attempt to examine the relationship between education management and students academic performance (Orodho, 2014; UNESCO, 1999; United Nations, 2013; Waweru & Orodho, 2014). The results reveal rather spurious relationship (Waweru & Orodho, 2014). According to Swami Vivekananda (1900) education is for life-building, man-making, character-building, assimilation of ideas, exposition of completed individuality and enkindling the urge of spirituality inherent in every mind.

Study habits, organizational climate, teachers teaching effectiveness and their job satisfaction and other factors play a very important role in the life of students. Success or failure of each student depends upon his own study habits, organizational climate, teachers teaching effectiveness and their job satisfaction etc. Of course, study is an art and as such it requires practice. Some students study more but they fail to achieve more. Others study less but achieve more. Success of each student definitely depends upon ability, intelligence, climate, culture and effort of students. No doubt, regular habits of study bring their own rewards in the sense of achievement of success. Freiberg and Stein (1999) described school climate as the heart and soul of the school and the essence of the school that draws teachers and students to love the school and to want to be a part of it. This renewed emphasis on the importance of school climate was further reinforced by a meta-analysis study performed by Wang *et al*. (1997), which found that school culture and climate were among the top influences in affecting improved student achievement. Their study also found that state and local policies, school organization and student demographics exerted the least influence on student learning.

According to Hoy and Tarter (1997), unhealthy schools are deterred in their mission and goals by parental and public demands. Unhealthy schools lack an effective leader and the teachers are generally unhappy with their jobs and colleagues. In addition, neither teachers nor students are academically motivated in poor schools and academic achievement is not highly valued. Healthy schools that promote high academic standards, appropriate leadership and collegiality provide a climate more conducive to student success and achievement (Hoy *et al*. 1990). The overwhelming majority of studies on school climate in the past have focused on teachers and leader– teacher relations and subsequent issues of job satisfaction. Miller stated 14 years ago that school climate has rarely been studied in relation to its effect on student achievement (Miller 1993). In recent years the emphasis on climate has shifted from a management orientation to a focus on student learning (Sergiovanni 2001).

So, the regression analysis is a popular and dependable methodology for determining the influences of factors on academic performance of students in many areas of research including educational research. The strength of regression analysis is the ability to capture direct effects or multiple relationships simultaneously, while providing a simple and fast estimation results. Some times the indirect factors play a significant role on academic performance of students. Thus, the regression analysis is not an appropriate method to determine the indirect effects of factors. Hence, in this article used the applications of Path Analysis or Structural Equation Modeling (SEM). SEM as a statistical technique has increased in popularity since it was first conceived by Wright (1918; 1934) a biometrician who developed the path analysis method to analyze genetic theory in biology. SEM enjoyed a renaissance in the early 1970s, particularly in sociology and econometrics (*Goldberger and Duncan 1973)* and later spread to other disciplines, such as psychology, political science, and education (Werts and Linn, 1970). It was believed that the growth and popularity of SEM was attributed to a large part to the advancement of software development that have made SEM readily accessible to substantive researchers who have found this method to be well-suited to addressing a variety of research questions (*MacCallum* and *Austin 2000)*. Therefore, the main purpose of this study was to use SEM to develop a model that offers a plausible explanation for direct and indirect relationships of factors on academic performance of students.

2. MATERIALS AND METHODS

2.1 Subjects

The study was conducted to determine the variables influential in determining the academic performance of secondary school students in Shiggaon taluka, Haveri District, Karnataka during September to November 2014. A total of 30 secondary schools were selected systematically from Shiggaon. Random samples of 30 secondary schools were selected and then random samples of 75 teachers and 300 students were selected from selected schools from all Shiggaon taluka. Before the start of the final study, we conducted a pilot study to examine the reliability of four tools or scales by taking a convenient sample of 60 students with 12 teachers. The reliability of the instruments was also ascertained using a test-retest method. The reliability coefficient of 0.8997, 0.9147, 0.9968 and 0.9758 were obtained respectively for the instruments using Pearson product-moment correlation coefficient. Finally, the tools on study habits was administered to students, teacher's teaching effectiveness, and organizational climate were and job satisfaction scales were administered to teachers. Then, they asked to read and fill the questionnaires. After receiving the questionnaires, scoring was made with maximum weight 5 and minimum weight is 1 given to teacher's teaching effectiveness, organizational climate and job satisfaction of teachers questionnaires, but maximum weight 4 and minimum weight is 1 was given to study habit scale for positive items. Similarly reverse scoring was to negative items. Further, for purpose of statistical analysis, total score was calculated in all four questionnaires. The overall response rate was 99.50%.

2.2 Selection of Variables

Taking into consideration the data available for the analysis, the academic performance of students (Y) is taken as a response variable and four factors were taken as explanatory variables. All four explanatory variables have been chosen as direct and indirect effective such as

X1=Study habits
X2=Organizational climate
X3=Teachers teaching effectiveness
X4=Job satisfaction

The numerical data were obtained for all these variables with sufficient care and response rate have been observed more than 99.50% in all teachers and students, it leads to good quality of analysis.

2.3 Statistical Analysis

The correlation matrix was generated using Pearson's 'r' to determine the relationships between and among explanatory variables with academic performance of students. The presence of explanatory variables with very strong relationships with response variable prevents an SEM solution. Statistical analysis was performed using SPSS for Windows Version 21.0 (SPSS; Chicago, IL, USA, 2012). Further analysis on SEM was performed using Analysis of Moment Structure Version 5.0.1 (AMOS: ADC, Chicago, IL, USA, 2003). The main focus was on testing hypotheses about relationships among the variables in the structural model. The criteria for assessing the structural model are the same as the measurement model using standardized regression estimates (B). Evaluation of structural model involves use of fit indices (Bentler and Chou 1987; Hu and Bentler 1999; Steiger 1990). The chi-square statistic provides a test of the null hypothesis, ensuring that the theoretical model fits the data. If p-values are more than 0.05 indicate a good fit. A root mean square error of approximation (RMSEA) value close to 0.06, and adjusted goodness-of-fit index (AGFI) and comparative fit index (CFI) values close to 0.9 indicate acceptable fit of the model. The AMOS also allows the use of modification indices to improve the model fit by drawing a correlation function between the identified variables (Byrne 2001; Cheah 2010). All the hypothesized paths in the conceptual model (Figure 1) is tested and included in the structural model. A statistical significance was set at 5% level of significance ($p<0.05$).

2.4 Results

The relationships between explanatory variables and predictor variable i.e. academic performance of students are shown in Table 1. It clears that, the significant and positive correlation was observed between academic performance of students and study habits of students of secondary schools ($r=0.6647$, $p<0.05$), with organizational climate of secondary schools of teachers ($r=0.8857$, $p<0.05$), with teaching effectiveness of teachers of secondary schools ($r=0.7303$, $p<0.05$) and with job satisfaction of teachers of secondary schools ($r=0.7724$, $p<0.05$) at 5% level of significance. The significant relationships among explanatory variables were also estimated (Table 2). It shows that a positive and significant relationships were observed ($p<0.05$) among all explanatory variables**.**

Table 1:
Correlation between academic performance of students with study habits, organizational climate, teachers teaching effectiveness and job satisfaction of teachers of secondary schools

Variables	Correlation between academic performance of students with		
	r-value	t-value	p-value
Study habits	0.6647	6.7763	0.00001*
Organizational climate	0.8857	14.5313	0.00001*
Teachers teaching effectiveness	0.7303	8.1410	0.00001*
Job satisfaction	0.7724	9.2619	0.00001*

*p<0.05

Table 2:
Correlations among four independent variables i.e. study habits, organizational climate, teachers teaching effectiveness and job satisfaction of teachers of secondary schools

Independent variables	r-value	t-value	p-value
Study habits vs Organizational climate	0.6962	7.3853	0.00001*
Study habits vs Teachers teaching effectiveness	0.6966	7.3935	0.00001*
Study habits vs Job satisfaction	0.6645	6.7727	0.00001*
Organizational climate vs Teachers teaching effectiveness	0.7524	8.6995	0.00001*
Organizational climate vs Job satisfaction	0.8027	10.2491	0.00001*
Teachers teaching effectiveness vs Job satisfaction	0.8003	10.1635	0.00001*

*p<0.05

2.5 The structural model for academic performance of students

A path diagram of the structural model of factors that influence academic performance of students is shown (Diagram 1). All paths that were significant or not significant are shown in the diagram except study habits and teachers teaching effectiveness. The RMSEA was 0.049, AGFI 0.85, and CFI 0.89, all of which indicated an acceptable fit. The chi-square value for the structural model was 14.0401, p=0.1170, with degrees of freedom of 3 (p>0.05). Therefore, the SEM model of factors that influence academic performance of students was significant. This model was acceptable and better fit model to determine the contributory relationships among factors determining academic performance of students.

2.5.1 Decomposition of Direct Effects

All of the direct paths hypothesized within SEM model were significant and in the expected direction (Table 3). There were also two paths i.e. organizational climate and job satisfaction of teachers were predicted and significant greater on academic performance of students. but, the paths of study habits and teachers teaching effectiveness were found to be not significant (p>0.05) (Table 3).

2.5.2 Decomposition of Indirect effects

There were numerous significant indirect effects between nonadjacent levels in the model Table 3). These total indirect effects represent all possible paths. Specific indirect effects have to be calculated by multiplication of the estimates of the direct effects involved in the total pathways. In summary, these were:

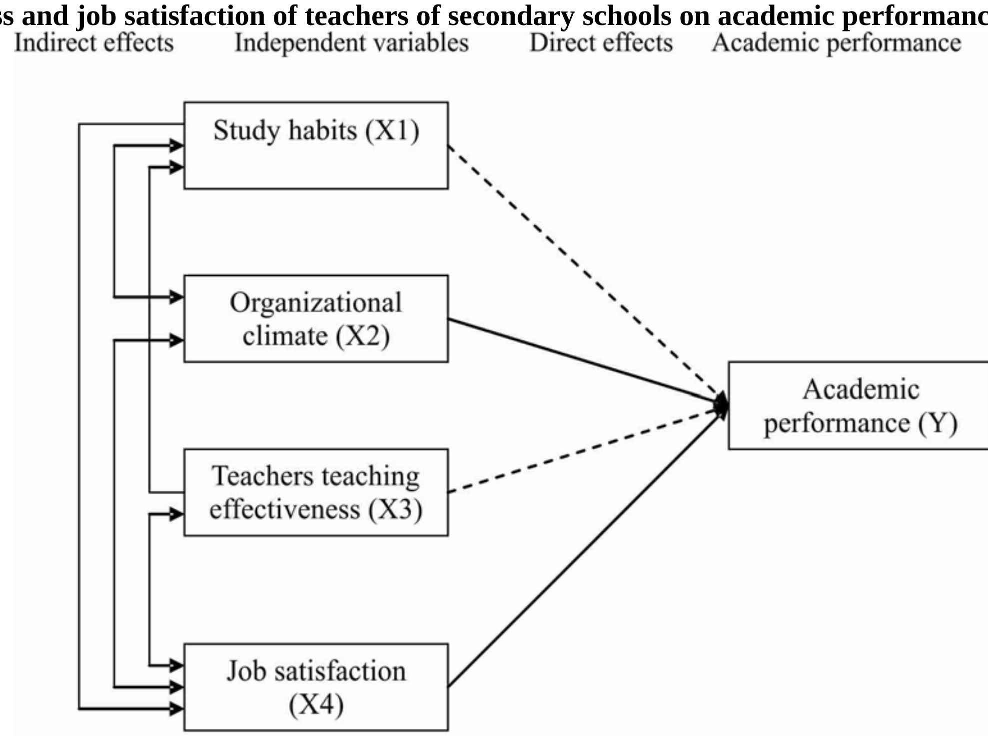
(i) Study habits → organizational climate (β=0.6961),
(ii) Study habits → Teaching effectiveness (β=0.6965),
(iii) Study habits → Job satisfaction (β=0.6645),
(iv) Organizational climate → Teaching effectiveness (β=0.7524),
(v) Organizational climate → Job satisfaction (β=0.8026),
(vi) Teaching effectiveness → Job satisfaction (β=0.8002),

Table 3:
Direct and indirect effects of study habits, organizational climate, teachers teaching effectiveness and job satisfaction of teachers of secondary schools on academic performance of students

Independent variables	Direct effects	Indirect effects through			
		X1	X2	X3	X4
Study habits (X1)	0.0749	-	0.3782*	0.4286*	0.1499
Organizational climate (X2)	0.7438*	0.2145*	-	0.2244	0.6530*
Teachers teaching effectiveness (X3)	0.0726	0.1938	0.1790	-	0.5726*
Job satisfaction (X4)	0.1833*	0.0426*	0.3274*	0.3599*	-

*p<0.05

Diagram 1:
The direct and indirect path coefficients of study habits, organizational climate, teachers teaching effectiveness and job satisfaction of teachers of secondary schools on academic performance of students



3. DISCUSSION

Analysis of relationship among study habits, organizational climate, teaching effectiveness and job satisfaction of teachers have significant and positive correlation with academic performances of students of secondary schools. This suggests that study habits, organizational climate, teaching effectiveness and job satisfaction of teachers could predict academic performances.

Further, in the study, study habit and organizational climate have significant direct effect on academic performances of students, but the direct effect of teaching effectiveness and their job satisfaction on academic performances of students is found to be not significant. However, the indirect effect of study habits of students through the organizational climate, teaching effectiveness and job satisfaction of teachers is positive and significant. Also indirect effect of organizational climate through the study habits of students, teaching effectiveness and job satisfaction of teachers is positive and significant. Similar finding were obtained for other two variables i.e. teaching effectiveness and job satisfaction of teachers through the study habits of students and organizational climate of secondary schools.

4. LIMITATIONS AND CONCLUSION

It is important to acknowledge that this study has some limitations. Although the sample size for this study was relatively big, the fact still remains that it did not represent the totality of teachers and students in the secondary schools in Shigoan, Haveri, Karnataka, India. Thus, a note of caution needs to be sounded when generalizing the study's findings. Despite this limitation, the findings of the study have provided a further need on how to improve upon the academics of students of secondary schools in the area. In particular, the study has shown that study habits, organizational climate, teaching effectiveness and job satisfaction of teachers cannot be over emphasized in academic success of students. For the present study concern, the study habits, organizational climate, teaching effectiveness and job satisfaction of teachers have positive correlation with academic performance of secondary school students.

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B-MODEL AND FM-MODEL IN STOCHASTIC PROCESSES

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ABSTRACT :
On the basis of well known B-model a cummulative damage model is introduced. The model is based on fracture mechanics hence named as FM-model. The objective of the paper is to relate the basic idea of B-model in such a way so as to transfer results from one structure to another. The accounts of the geometry of structure is taken through stress intensity fator. The model is applicable in wider sense both deterministic as well as to random loading.
Keywords: ***Random fatigue, B-Model, fracture mechanics, FM-Model, numerical cumulative damage model, Stochastic load process.***
MSC Code : 60GXX

INTRODUCTION

Varying loads acting on a structure will cause initiation and propagation of cracks. **The cumulative damage (CD)** is defined as the irreversible accumulation of damage through lifetime, which ultimately causes failure. The process is random and justifies reduction of the reliability of the structure.

A mathematical model which is primarily based on physical observable quantities is necessary to describe the CD-process. Usually, distinction is made between two main groups of damage models: **deterministic models and probabilistic models.** Deterministic models only give information about the mean damage accumulation, thus ignoring the fluctuations characteristic for fatigue. The use of a probabilistic model makes it possible to take account of the fluctuations and making the CD-model more realistic.

First, a proper probabilistic model must be chosen. Secondly, the model parameters must be determined. The number of model parameters depends on the complexity of the phenomenon considered. Thirdly, typical progresses of the CD-process must be established by simulations. Hereby, the sample functions are determined. A sample function is an event of a random process, in this case it is a function which describes the accumulation of damage in a component as a function of time. Discretizing the time, the damage also has to be discretized, and a measure of damage must be chosen. Finally, the probabilities for given events and the statistical moments can be estimated using the sample functions. [1]

Generally, any parameter can be used as a damage measure. The probabilistic, cumulative damage model described only requires that the damage measure describes a non-decreasing function. In the model described by the crack increment Δa is measured for fixed values of the increment in numbers of load cycles, ΔN, i.e., Δa is random. [7] The cumulative damage model developed in this paper, uses Δa as a fixed parameter whereas ΔN to cause Δa is random. [2] This is the most appropriate method because the distribution of numbers of cycles performed to reach a given crack length a, can hereby be determined. Further, Δa can be regarded as a material constant.

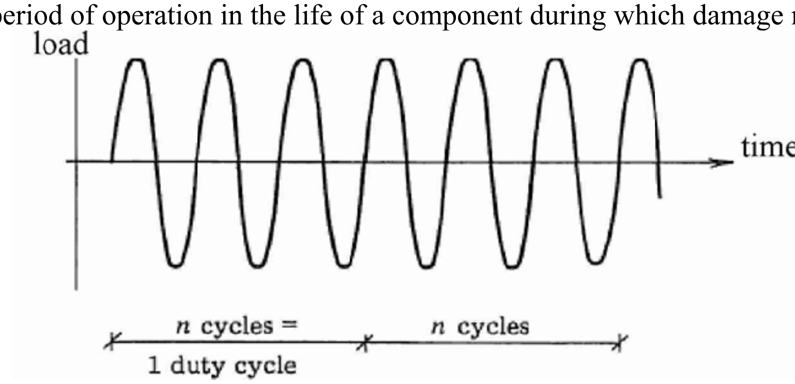
NOTATIONS

a - crack length
a_0 - initial crack length
a_b - failure crack length
$\delta a, \Delta a$ - crack length increment
b - failure state

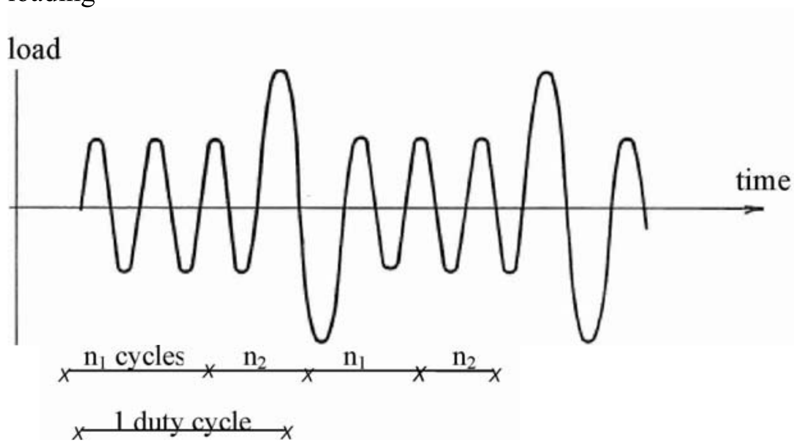
C	- material constant
CD	- cumulative damage
d	- damage state
da	- crack length increment
DC	- duty cycle
dN, ΔN	- increase in numbers of cycles
fx	- probability mass function
K	- stress intensity factor
K_{max}	- maximum stress intensity factor
K_{min}	- minimum stress intensity factor
ΔK	- stress intensity factor range
ΔK_{eff}	- effective stress intensity factor range
M	- material constant
N	- numbers of cycles
δN	- increases in numbers of duty cycles
$\bar{p}_0$	- initial state vector
p	- (1 – q)
$\bar{p}_x$	- state vector
$\bar{\bar{P}}$	- transition matrix
q	- transition probability
x	- time
X	- random variable
γ_0	- initial crack growth rate
λ	- numbers of load cycles in one duty cycle
μ	- mean value
π_j	- probability that damage is initially in state j
$\Delta\sigma$	- stress range
$\Delta\sigma_{eff}$	- effective stress range
σ_{cl}	- crack closure stress

1 THE B-MODEL

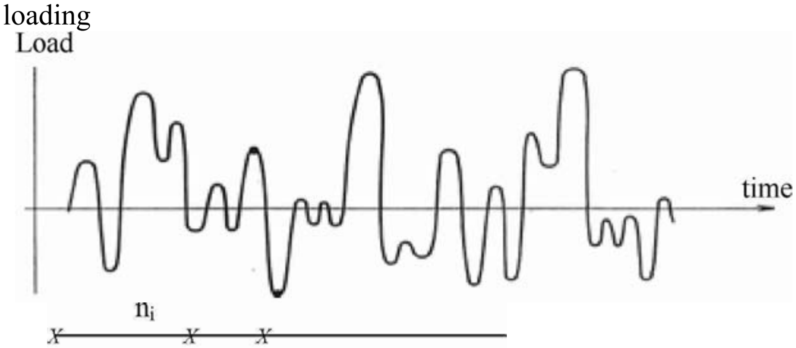
A basic element in the B-model is the division of the load into duty cycles, see figure 1.1. A **duty cycle (DC)** is defined as a repetitive period of operation in the life of a component during which damage may accumulate



(a) Constant-amplitude loading



(b) Variable-amplitude loading



n_i duty cycle = 1 duty cycle; i = 1, 2, …

(c) Random loading

Figure 1.1: Examples of duty cycles.

The discrete time, *x,* is measured in numbers of DC's, $x = 1, 2, ...$ The damage accumulation is considered as a stochastic process in which the possibility of damage accumulation is present each time the structure has experienced a DC.

The damage *d* is assumed to be discrete with the states $d = 0, 1, 2, ..., b.$ where *b* corresponds to failure. Further, it is assumed that the increment of damage at the end of the DC only depends on the DC itself and the state of damage present at the start of the DC. The damage only increases by one unit at a time.

Hereby, the damage accumulation process can be regarded as a discrete-time, discrete-state Markov process. [5]

Such a Markov process is completely described by its transition matrix (one transition matrix for each duty cycle) and by the initial conditions.

The initial probability distribution of the damage states is given by the vector

$$\overline{P}_0 = \{\pi_0, \pi_1, \pi_2, ..., \pi_{b-1}, \pi_b\} \quad \pi_j \geq 0 \qquad (1.1)$$

where

π_j = prob {damage is initially in state j} $j = 0, 1, 2, \dots, b-1$

$\boldsymbol{\pi_b} = 0$

Thus, the π_j - values form the probability mass function of the initial damage state. The distribution of the initial crack lengths in a weld can be given as an example.

As mentioned earlier, the damage only increases by one unit at a time. Thus, it is possible to establish the transition matrix $(b \times b)$ for the i^{th} duty cycle $\overline{\overline{P}}_i$ given by

$$\overline{\overline{P}}_1 = \begin{pmatrix} p_0 & q_0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & p_1 & q_1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & p_2 & q_2 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & p_j & q_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad i = 1, 2, \dots \qquad (1.2)$$

where the conditional probabilities

p_j= prob {remains in state j | previously in state j}

$j = 0,1,2,...,b$ (1.3)

q_j = prob {goes to state j + 1 | previously in state j}

and

$p_j \geq 0$

$q_j \geq 0$ $\quad j = 0,1,2,...,b$

$p_j + q_j = 1$

The damage state at the time x is then given by the vector

$$\overline{P_x} = \overline{P_0}\,\overline{\overline{P}}_1\,\overline{\overline{P}}_2 \dots \overline{\overline{P}}_x \qquad (1.4)$$

where

$\overline{P_0}$ is given by (1.1)

$\overline{P_x}$ = {p_x (j)} = prob {damage is in state j at time x}

If $\overline{\overline{P}}_1$ is identical for all duty $\overline{\overline{P}}_1 = \overline{\overline{P}}$ cycles, i.e. = $\overline{\overline{P}}_i = \overline{\overline{P}}$, (1.4) reduces to

$$\overline{p_x} = \overline{p_0}\,\overline{\overline{P}}^X \qquad (1.5)$$

The advantage of the B-model is that once the model parameters π_j, p_j ; q_j, j = 0, 1, 2,...,b are determined, the state of damage in the given structure is available at any time using (1.4). This means that all statistical information about the damage process can be represented by the model.

2. THE BASIC IDEAS OF THE FM-MODEL

The purpose of this paper is to relate the basic ideas of the B-model described chapter1 in to physical problems in such a way, that it is possible to transfer results from one structure to another. This description is named the **Fracture Mechanical Model (FM-model).**

The crack length, a, can be used as a damage measure which is advantageous since a is a quantity that can be observed. Further, a reliable connection between the loading process and the process of fatigue, in the form of crack growth, has been established and verified by experiments. The connection is represented by a crack propagation law, which in most cases expresses the crack propagation as a function of the so-called stress intensity factor range ΔK. This is a fracture mechanical value defined by the stress field near a crack tip. [8]

In the FM-model, the damage is assumed to progress in steps of the length δa. Hereby, δa is assumed to be a material constant. Thus, the state of damage can be defined as

$$a_j = a_0 + j\,\delta a \quad j = 0, 1, 2, ..., b \qquad (2.1)$$

Where

a_j = crack length at damage state j [mm]

a_0 = initial crack length [mm]

a_b = failure crack length [mm]

The above has the effect that both the damage process and the crack growth process are stepwise even though a continuous process for the latter would be more realistic from a continuum point of view. Bearing in mind that the material is not a continuum but is inhomogeneous, from a micro point of view, the model seems reasonable.

In a given crack state, $a = a'$, the propagation of the crack is assumed to be described by a Bernoulli random variable X, [1]

$$X = \begin{cases} 0 & \text{the crack remains in the given state the} \\ 1 & \text{crack propagates } \delta a \end{cases} \tag{2.2}$$

The probability mass function of X then is

$$f_x(x) = \begin{cases} p_j = 1 - q_j & \text{for } X = 0 \\ q_j & \text{for } X = 1 \end{cases} \qquad j = 0,1,2,\ldots\ldots b \tag{2.3}$$

The quantity q_j is known the transition probability, see also (1.3). The crack growth problem is then reduced to the determination of $q_j = q(\Delta K_j) = q(\Delta K(a_j))$.

The most simple situation occurs if ΔK is constant, i.e. the crack tip loading is constant no matter how long the crack is. If so, $\Delta K_j = \Delta K$ and hence, $q_j = q$.

Combining the empirical Paris law, which is one of the most frequently used crack propagation laws, and the binomial distribution, the estimation of q is possible.

Paris law [10], is given as

$$\frac{da}{dN} = C(\Delta K)^m \tag{2.4}$$

where

da = increase in crack length [mm]
dN = increase in numbers of cycles
C = material constant [mm / (MPa$\sqrt{m}$)^m]
m = material constant
$\Delta K = K_{max} - K_{min}$ = stress intensity factor range [MPa$\sqrt{m}$]
K_{max} = maximum stress intensity factor in one load cycle [MPa$\sqrt{m}$]
K_{min} = minimum stress intensity factor in one load cycle [MPa$\sqrt{m}$]

Introducing the step length δa, (2.4) becomes

$$\frac{\delta a}{E(\delta N)} = \lambda C (\Delta K)^m \tag{2.5}$$

where

$E(\delta_N)$ = the expected value of the random variable δN corresponding to the expected numbers of duty cycles which is used to pro-pagate the crack one step δa.
λ = numbers of load cycles in one duty cycle

Every time the crack tip is exposed to one duty cycle, the same trial is repeated. Thus, the expected numbers of duty cycles is given as the first moment in the geometric distribution. [1]

$$E(\delta N) = \sum_{\delta n=1}^{\infty} \delta n \, q(1-q)^{\delta n - 1} = \frac{1}{q} \tag{2.6}$$

Insertion of (2.6) into (2.5) leads to

$$q = \frac{\lambda C}{\delta a} (\Delta K)^m \tag{2.7}$$

This means, that for a given material the transition matrix (2.2) is determined. The damage states in structures, made of the given material, are then calculated using (2.5). Account of the geometry of the structure is taken through the stress intensity factor range ΔK.

3. RANDOM LOADING

The loading on most structures is of random character, which is more difficult to handle with than deterministic loading.

Due to the randomly varying loading, the stress intensity factor will also vary randomly. The value of the stress intensity factor range, ΔK, depends on the definition of a load cycle.

A realization of a stochastic load process is shown in figure 3.1. Half a stress range is defined as the part of the realization between two adjacent points of reversal, i.e.

$$\frac{1}{2}\Delta\sigma = \sigma_1 - \sigma_2 \qquad (3.1)$$

where

$\Delta\sigma$ = stress range [MPa]
σ_1 = maximum stress [MPa]
σ_2 = minimum stress [MPa]

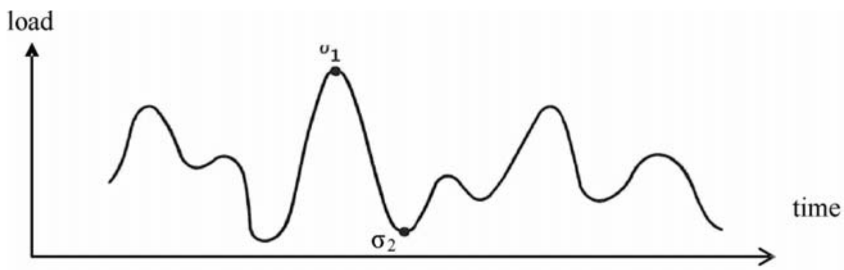


Figure 3.1: Realization of a stochastic load process. σ_1 and σ_2 correspond to the maximum stress and the minimum stress, respectively, in the forthcoming half load cycle.

The FM-model itself does not take into account the well-known effects of acceleration and retardation. This can be done using a crack closure model when ΔK is calculated.

Considering the realization in figure 3.1, the damage increment in the next half cycle will depend on the loading history, the geometry of the structure and the extreme values σ_1 and σ_2. The progress of the effective stress intensity factor, corresponding to the realization in figure 3.1, is shown in figure 3.2.

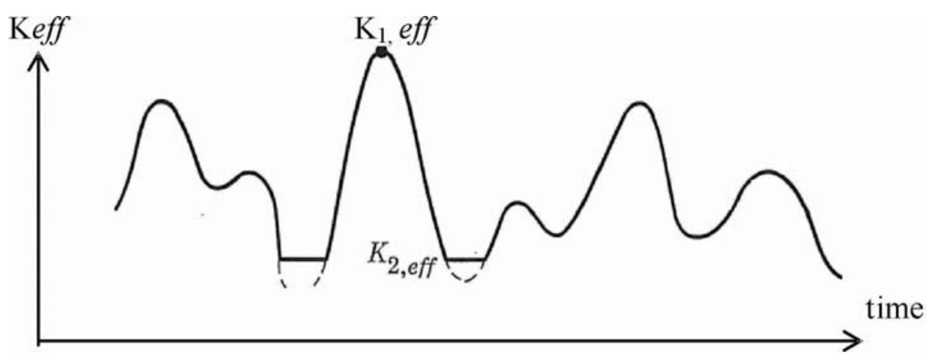


Figure 3.2: Progress of effective stress intensity factor, K_{eff}, corresponding to the realization in figure 3.1. $K_{1,eff}$ and $K_{2,eff}$ correspond to the maximum and the minimum effective stress intensity factor, respectively, in the forthcoming half cycle.

Introducing the effective stress intensity factor range,

$$\Delta K_{eff} = \Delta\sigma_{eff} \, F\sqrt{\pi a} = (\sigma_{max} - \sigma_{min}) \, F\sqrt{\pi a} \qquad (3.2)$$

where

$\Delta\sigma_{eff}$ = effective stress range [MPa]
σ_{max} = maximum stress [MPa]

σ_{cl} = crack closure stress [MPa]
F = geometry function
a = crack length
Paris law (2.4) becomes

$$\frac{da}{dN} = C\,(\Delta K_{eff})^{m} \tag{3.3}$$

The transition probability changes to

$$q_j = \frac{\lambda C}{\delta a}\,(\Delta K_{eff}(a_j))^{m} \tag{3.4}$$

whereas the expected value and variance of the numbers of duty cycles to propagate the crack one step from a_j to $a_j + \delta_a$,

$$E\,[\delta N_j] = \frac{1}{q_j} \tag{3.5}$$

$$Var\,[\delta N_j] = \frac{1-q_j}{q_j^2} \tag{3.6}$$

The total numbers of duty cycles applied to the structure to propagate the crack to the crack length a_j, and the expected value are also unchanged.

$$N = \sum_{k=0}^{j-1} \delta N_k \tag{3.7}$$

$$E\,[N] = \sum_{k=0}^{j-1} \frac{1}{q_k} \tag{3.8}$$

The approximate value of (3.8) becomes

$$E\,[N] \simeq \frac{1}{\lambda C}\int_{ao}^{a} (\Delta K_{eff}(a))^{-m}\,da \tag{3.9}$$

The variance is still given as

$$Var\,[N] = \sum_{k=0}^{j-1} \frac{1-q_k}{q_k^2} \tag{3.10}$$

with as the approximative value,

$$Var\,[N] \simeq \frac{\delta a}{\lambda^2 C^2} f^2(a) - \frac{1}{\lambda C} f(a) \tag{3.11}$$

but, where

$$f(a) = \int_{ao}^{a_j} (\Delta K_{eff}(a))^{-m}\,\delta a \tag{3.12}$$

Thus, the FM-model is also available if the loading is random.

4. CONCLUSION

On the basis of the well-known B-model, a new numerical cumulative damage model based on fracture mechanics is introduced. This model, the FM-model described in incorporates existing physical knowledge of crack propagation.

The cumulative damage is described by a discrete-time, discrete-state Markov process. The time is measured in numbers of duty cycles, whereas the state of damage is given as a crack length. The crack is assumed to propagate in steps of the length δa.

The Markov assumption is reasonable for small values of the step length, δa. Further, Paris law is used as crack propagation law.

One of the main results is that the step length δa appears to be a material constant so that, once the materials constants in Paris law, C and m, and δa are determined, the state of damage in any structure of the given material can be calculated numerically

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REVIEW ON WORD SENSE DISAMBIGUATION TECHNIQUES

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ABSTRACT :

All the natural languages contain polysemous word. Word sense disambiguation (WSD) refers to the computational identification of the correct sense of a word and is considered an AI complete problem. WSD is an important task in many natural language applications. In this paper, we review existing techniques for word sense disambiguation.

1 INTRODUCTION

Word Sense Disambiguation (WSD) refers to the task of assigning the correct sense of a polysemous word in a given context. All natural languages contain polysemous words, i.e., words having multiple senses (meaning). For example,

The noun still has following 4 senses (first 1 from tagged texts) listed in English WordNet 2.1.

1. (2) still -- (a static photograph (especially one taken from a movie and used for advertising purposes); "he wanted some stills for a magazine ad")
2. hush, stillness, still -- ((poetic) tranquil silence; "the still of the night")
3. still -- (an apparatus used for the distillation of liquids; consists of a vessel in which a substance is vaporized by heat and a condenser where the vapor is condensed)
4. distillery, still -- (a plant and works where alcoholic drinks are made by distillation)

This is true of every language. Human beings are able to disambiguate these words almost effortlessly. However, identifying the most appropriate sense of a word computationally is quite difficult.

WSD is an "intermediate task" for many NLP systems, including machine translation, information extraction, information retrieval etc. It is essential for Natural Language Understanding applications. WSD can be viewed as a classification task where each occurrence of a word is assigned one or more sense labels using the local context or external knowledge sources. Two variants of generic WSD task exists: Lexical Sample and All-words WSD. In lexical sample (or targeted WSD) a system is required to disambiguate a restricted set of target words. In All-words WSD, a system is expected to disambiguate all open-class words in a text (i.e. nouns, verbs, adjectives and adverbs). WSD has been an area of research interest since 1950's. A number of techniques have been developed since then. The existing WSD techniques can be broadly categorized into Knowledge-based (or dictionary-based) and corpus-based techniques. However, this strict classification does not hold for many of the algorithms including Yarowsky (1995), which combines features from both. Dictionary-based techniques utilize information from external knowledge sources such as Machine Readable Dictionaries (MRDs), thesauri or ontology. WordNet is the most popular resource used in WSD research. Corpus-based techniques rely on information extracted from a large sense-tagged corpus. The information extracted from the corpus includes distributional information, context, and further knowledge that has been annotated in the corpus or added during pre-processing. The data acquisition

bottleneck is the major difficulty of a corpus-based approach. Creating sense tagged corpus is time consuming. An alternative is to use bootstrapping or unsupervised approaches which are less demanding in terms of tagged data. In this paper, we review some of these techniques. Rest of the paper is organized as follows:
In Section 2, knowledge-based techniques are discussed. In section 3, we review corpus-based techniques. Section 4 discusses open issues and finally we conclude in section 5.

2 KNOWLEDGE-BASED TECHNIQUES

The knowledge-based techniques utilize knowledge extracted from machine Readable Dictionaries (MRDs), Thesauri or Computational Lexicon to infer the senses of words in context. MRDs provide information about word senses which has been utilized for WSD by many researchers including [1][4]. A thesaurus provides a rich network of word associations and a set of semantic categories, which can be utilized in WSD and for large scale language processing. The work has been carried out using WordNet[8][10][12], Longman Dictionary of Contemporary English (LDOCE) [4] and Roget's International Thesaurus [7].

Lesk [1] performed one of the early works in WSD. He used the number of words overlapping between the sense definitions of ambiguous words and the definition of context words surrounding the ambiguous word in a given text. The sense, maximizing the overlap between the context and sense definitions was the considered the most likely sense. Lesk's work served as the basis for most of the subsequent dictionary-based WSD systems. Lesk's algorithm fails to disambiguate a number of ambiguous words if no co-occurrence was found between the context and sense definitions. Walker [2] attempted to handle this problem using subject codes provided by Roget's thesaurus. The underlying assumption was that the subject codes assigned to a word reflects the semantic category of the word. Several extensions to Lesk's algorithm have been proposed. Guthrie et al. [4] used the subject-dependent co-occurrence neighborhood to improve disambiguation results. They used subject classifications given in LDOCE for this purpose. Another improvement to Lesk' algorithm was proposed by Cowie et al. [5] using simulated annealing algorithm. Vasilescu et al.[29] performed comparative evaluation of varaint's of Lesk algorithm and found simplified Lesk better than original Lesk.

Yarowsky [7] extracted classes of words using the explicit concept hierarchy of Roget's thesaurus and used this information in disambiguation. More recently, WordNet has been used by a number of researchers in disambiguation task. Banerjee and Pederson [23] proposed adapted Lesk algorithm. They used WordNet semantic relations such as hypernym, hyponym, meronym, tryponym and attribute of each word glosses in disambiguation. In a subsequent work, Banerjee and Pederson [25] proposed a new measure of semantic relatedness which extends the glosses of the concept by including the glosses of other concepts to it is related in the WordNet concept hierarchy. They proposed a WSD algorithm using extended gloss overlap. The experimental evaluation done on SENSEVAL-2 lexical sample data suggests that their algorithm more accurate than all but one of the participating systems. Patwardhan et al. [26] evaluated a variety of semantic relatedness measures including Resnik [10], Jiang and Conrath [16], Lin [15], Leacock and Chodorow [18] and Hirst and St. Onge [19]. The adapted Lesk and the semantic distance measure of Jiang and Conrath [16] resulted in the highest accuracy on the Senseval-2 English lexical sample data.

Baldwin *et al.* [32] proposed a new method of MRD-based word sense disambiguation using definition expansion via ontology. They experimented with character- and word-based tokenization and a range of lexical relations. Khapra *et al.* [31] studied domain specific WSD for nouns, adjectives and adverbs for English, Hindi and Marathi. They used dominant senses of words in specific domains for performing disambiguation. Singh and Siddiqui [35] evaluated the effect of context window size, stemming and stop word on Hindi WSD task. They used direct overlap between context and sense definitions obtained from Hindi WordNet for disambiguating a polysemous word. The work in Singh & Siddiqui [38] used Lesk-based algorithm to evaluate the role of semantic relations from Hindi WordNet on Hindi WSD. The baseline corresponds to sense definition extended by adding example sentences and synonyms to its gloss. An improvement of 12.09% in precision over the baseline was reported for the case when all relations (hypernym, hyponym, holonym and meronym relations) are considered. The maximum increase of 9.86% in precision by a single relation was observed using hyponym. In [36], Hindi WordNet hierarchy is used to take advantage of semantic similarity in the disambiguating Hindi nouns. They used Leacock-Chodorow [18] semantic relatedness measure and obtained significant improvement over direct overlap-based method.

3 CORPUS-BASED TECHNIQUES

There are two possible approaches to corpus-based WSD systems: supervised WSD and unsupervised WSD. Supervised approaches use machine-learning algorithm to learn a classifier for disambiguation. The unsupervised approaches use raw corpus but suffer from low accuracy.

3.1 Supervised Techniques

Supervised WSD use some machine-learning algorithm to learn a classifier from a sense-annotated corpus automatically. The existing WSD works have used decision list, decision tree, Naive-Bayes (NB), Neural Network, Support vector machine, etc. for disambiguation. Decision lists is one of the most efficient supervised algorithms which learns a list of rules by extracting features from a training corpus and apply these rules to identify correct sense of a word. The commonly used features include part of speech (POS), collocation vector, neighboring words and their POS's, co-occurrence vector, etc. These features are used to create rules of the form (feature value, sense, score) which constitute the decision list.

NB classifiers have been used extensively utilized in WSD [14] [22] [27] and have been proven quite effective in WSD task. Mooney [14] experimentally compared seven different learning algorithms for the problem of learning to disambiguate from context. In experimental evaluation simple Bayesian and neural-network methods, were shown to perform better than alternative methods such as decision-tree, rule-based, and instance-based techniques on the problem of disambiguation. Le and Shimazu [27] used rich knowledge, represented by ordered words in a local context and collocations in their NB classifier and achieved 92.3 % accuracy for four common test words (interest, line, hard, serve). In [22] an ensemble of Naive Bayesian classifiers is used which uses co-occurring features of varying size. This simple classifier achieved an accuracy that was comparable to best previously published results on *line* and *interest* corpus. Towell and Voorhees [20] used a combination of two feed forward network to form a contextual representation that is used for disambiguation. The neural networks separately extract topical and local contexts of a target word from a set of sample sentences that are tagged with the correct sense of the target word.

Gale et al. [6] used k-NN classifier for disambiguating six ambiguous nous in Hansard corpus. K-Nearest Neighbor (kNN) works by learning context in the training set. During testing the test instance is matched with the learned context and k most similar context to it in the training set are selected. Each of these contexts is then assigned a score the sense having maximum score is assigned to the test instance. Gale achieved an accuracy of 90% with this simple classifier. Rezapour et al. [33] proposed a supervised learning algorithm for WSD based on k-NN. In order to improve accuracy they used a heuristics to weight the extracted features. Instead of giving similar weight to all features the heuristics give more weight to features that are important for disambiguation. It attempts to capture the importance of features in disambiguating a word based on its occurrence frequency. They used TWA (WWW.cse.unt.edu/~rada/downloads.html) corpus and used 5-fold cross-validation to estimate the performance of the algorithm. When compared with other existing corpora-based method proposed in [7] [3] [9] [13] their method outperformed all but Dagan & Itai [9]. Singh et al. [37] evaluated NB-classifier using 11 different features on a dataset consisting of 60 polysemous Hindi nouns[34]. They obtained a maximum accuracy of 86.1% when the base form of nouns appearing in the context was used as features.

Support Vector Machines (SVMs) are based on the idea of learning a hyperplane from a set of the training data so that it separates positive and negative examples and is located in the hyperspace in a manner that maximizes the distance between the closest positive and negative examples (called support vectors). SVM has shown to achieve the best results in WSD compared to several supervised techniques [21] [24].

3.2 Semi-supervised and Unsupervised Techniques

Unlike supervised algorithms, semi-supervised techniques small amount of seed instances for training. Most important contribution in this direction was by Yarowsky's [11] using decision list for disambiguation. His algorithm starts with some training examples representative, called seed set, for each sense These examples are extracted using seed collocations that are strong indicator of a particular sense. Then a supervised algorithm is used to identify collocations within the user-specified window that reliably partition the seed training data. These collocations are ranked using the log-likelihood ratio. The set of collocations having log-likelihood ratio higher than a set threshold constitute the decision list. The resulting classifier is then applied on the entire sample set to tag additional instances and new collocations are extracted from the tagged instances. The process is iterated until convergence. exploited two powerful properties of human language Yarowsky additionally applied two important properties of natural languages to tag remaining instances:

1. One sense per discourse: The sense of a target word is highly consistent within any given document or discourse.
2. One sense per collocation: Nearby words provide strong and consistent clues to the sense of a target word, conditional on relative distance, order and syntactic relationship.

A completely unsupervised sense disambiguation algorithm can only discriminate word senses without assigning sense tags. These algorithms require manual evaluation, as the sense clusters derived by these algorithms may not match the actual senses. The existing unsupervised WSD works are based on contextual clustering, word clustering [15] or graph-based approaches. Lin [15] clustered two words if they share some syntactic relationship. If in a context w_1, w_2, w_3, …, w_n represents the content words and w represent the target word, then the similarity between w and w_i is determined by the information content of their syntactic features. Co-occurrence graph based clustering uses graphs instead of vectors. Nodes in the graph correspond to words and the edges correspond to syntactic relation between them. One notable algorithm using co-occurrence graph based approach is Hyperlex, proposed by Veronis [28]. In this algorithm first a co-occurrence graph is built, such that the nodes are words occurring in the paragraphs of text corpus and edges between a pair of words are added to the graph if they co-occur in same paragraph. The relative co-occurrence frequency of the two words connected by an edge is used to assign weight to that edge. Then an iterative algorithm is applied and the node with highest relative degree is selected as a hub. The hubs are linked to the target word with zero weight edges and the minimum spanning tree (MST) of the entire graph is computed. The MST is used to disambiguate instances of the target word.

4 OPEN ISSUES

One of the major issues in WSD task is representation of word senses. An enumerative lexicon is widely adopted for an objective assessment of WSD systems. The major issue with enumerative approach is the division of senses. The sense distinctions in most dictionaries are often too fine-grained for most NLP applications[30]. WordNet, which is widely adopted within the NLP community, has very fine-grained sense divisions. The sense granularity problem has been an area of interest in the tasks of coarse-grained lexical sample and all-words WSD organized at Semeval-2007.

The knowledge acquisition bottleneck is another important issue in WSD as all WSD methods rely on use of knowledge provided either by corpora or dictionaries. A number of techniques for alleviating this problem have been proposed. These include bootstrapping, active learning, the automatic acquisition of training corpora, the use of cross- lingual information and the automatic enrichment of knowledge resources. Bootstrapping is one of the most promising research lines to open the knowledge acquisition bottleneck in WSD because of the unavailability of annotated data and availability of huge amount of unannotated data. Co-training [17] is an alternative algorithm applied to Classify Web pages. Co-training provides two views of the problem and apply at each learning iteration one of them alternatively. Each view is a learning classifier focusing on a set of features. The feature set for both views are mutually exclusive. Transductive learning is a bootstrapping approach consisting of introducing the unlabeled data in the learning process. It propagates by learning by the usual inductive approach from labeled data and then separating hyperplane by guessing the class of the unlabeled examples. The automatic enrichment of knowledge resources, especially of machine-readable dictionaries and computational lexicons is a new trend to overcome the knowledge acquisition bottleneck. A huge amount of work on the enrichment of knowledge resources has focused on the use of corpora for extracting collocations and relation triples of different kinds. For enriching existing WordNet with new semantic relations, collocations and relation triples need to be disambiguated. The large-scale efforts in knowledge acquisition and enrichment of knowledge resources will enable wider coverage and more accurate WSD systems.

5 CONCLUSION

In this paper, we surveyed past research associated with WSD. Both the knowledge-based and corpus-based techniques have been discussed. We also discussed the open issues in WSD research.

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ON NEW WEIGHTED DIFFERENCE-DIVERGENCE MEASURES THEIR INEQUALITIES, CONCAVITY AND NON-SYMMETRY

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ABSTRACT :
In the present communication, we have considered new symmetric weighted difference-divergences. Exploiting Csiszar's f-divergence for weighted distribution and calculus , we have established some new inequalities among then as such ;

(i) $$\alpha_5(P\|Q;W)\le\alpha_6(P\|Q;W)\le\frac{1}{3}\alpha_{10}(P\|Q;W)\le\frac{1}{2}\alpha_9(P\|Q;W) \quad (1)$$

And

(ii) $$\alpha_7(P\|Q;W)\le\frac{1}{3}\begin{cases}\alpha_{12}(P\|Q;W) & \le\frac{1}{2}\alpha_9(P\|Q;W)\\ \alpha_{13}(P\|Q;W)\end{cases} \quad (2)$$

Keywords: ***Weighted Triangular Divergence, Weighted Hellinger Divergence, Weighted Symmetric Chi-Square Divergence, Weighted f- divergence.***
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1.1 INTRODUCTION

Recently there is lot of interest in studying different types of divergence measures viz. symmetric and non-symmetric, logarithmic and non- logarithmic, weighted or non-weighted, probabilistic or non-probabilistic (Fuzzy). Kullback-Leibler [8] initiated the studies after the origin of Kerridge [7] inaccuracy. Csiszar's [2] f-divergence extended the studies further and Dragomir et al. [3-4] made a vital contribution in terms of inequalities and approximations space. Burbea and Rao [1] initially, in 1982 opened the chapter of convexity of divergence measures. From the last ten years Taneja [10] has enhanced the study of divergence measures, symmetric and non-symmetric, generalized and unified in a systematic way. Here in this communication, we are to enlarge the radius of studies, considering the importance / utility / weight of the strategy to achieve the goal or the target for decision making purpose. Earlier Kapur [6] has made it clear that every probability distribution is a function of weighted distribution. So why not to consider the weighted distribution corresponding to probability distribution likewise Guiasu's weighted entropy corresponding to Shannon's entropy.

Therefore, we consider the following information scheme for the purpose to define weighted new symmetric and non-symmetric divergence measures:

$$I.S.=\begin{bmatrix}E_1 & E_2 & E_3 & \dots & E_n\\ p_1 & p_2 & p_3 & \dots & p_n\\ q_1 & q_2 & q_3 & \dots & q_n\\ w_1 & w_2 & w_3 & \dots & w_n\end{bmatrix}=\begin{bmatrix}E\\P\\Q\\W\end{bmatrix} \quad (1.1)$$

where

$E=\{E_1,E_2,E_3,...,E_n\}$, set of events

$P=\{p_1,p_2,p_3,...,p_n\}$, set of probabilities

$Q=\{q_1,q_2,q_3,...,q_n\}$, set of revised probabilities

$W=\{w_1,w_2,w_3,...,w_n\}$, set of weights corresponding to probabilities

$$\sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} q_i, \qquad p_i, q_i \geq 0, \ \forall i = 1,2,....,n$$

and

$$w_i > 0, \qquad \forall i = 1,2,...,n.$$

The weighted divergence for the information scheme (1.1) can be defined as:

$$K(P\,||\,Q;W) = \sum_{i=1}^{n} w_i p_i \log \frac{p_i}{q_i} \quad (1.2)$$

Recently many researchers studied different types of divergence measures, parametric and non-parametric, symmetric and non-symmetric. Taneja [10] has considered the following symmetric and non-symmetric divergence measures with their adjoints viz.

SYMMETRIC DIVERGENCES

S.No	Divergence	S.No.	Adjoint
1.	$h(P\,\|\|\,Q) = \frac{1}{2}\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2$	1.	$h(Q\,\|\|\,P) = \frac{1}{2}\sum_{i=1}^{n} (\sqrt{q_i} - \sqrt{p_i})^2$
	Hellinger Discrimination		
2.	$\Delta(P\,\|\|\,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}$	2.	$\Delta(Q\,\|\|\,P) = \sum_{i=1}^{n} \frac{(q_i - p_i)^2}{q_i + p_i}$
	Triangular Discrimination		
3.	$\psi(P\|\|Q) = \chi^2(P\|\|Q) + \chi^2(Q\|\|P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i}$	3.	$\psi(P\|\|Q) = \chi^2(Q\|\|P) + \chi^2(P\|\|Q) = \sum_{i=1}^{n} \frac{(q_i - p_i)^2 (q_i + p_i)}{q_i p_i}$
	Symmetric Chi-square Divergence		
4.	$J(P\|\|Q) = K(P\|\|Q) + K(Q\|\|P) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}$	4.	$J(P\|\|Q) = K(Q\|\|P) + K(P\|\|Q) = \sum_{i=1}^{n} (q_i - p_i) \log \frac{q_i}{p_i}$
	J-Divergence (Jeffreys)		
5.	$I(P\|\|Q) = \frac{1}{2}[F(P\|\|Q) + F(Q\|\|P)] = \frac{1}{2}\left[\sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i + q_i}\right) + \sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i + q_i}\right)\right]$	5.	$I(P\|\|Q) = \frac{1}{2}[F(Q\|\|P) + F(P\|\|Q)] = \frac{1}{2}\left[\sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{q_i + p_i}\right) + \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i + q_i}\right)\right]$
	Relative Jensen-Shannon Divergence		
6.	$T(P\|\|Q) = \frac{1}{2}[G(P\|\|Q) + G(Q\|\|P)] = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right)$	6.	$T(P\|\|Q) = \frac{1}{2}[G(Q\|\|P) + G(P\|\|Q)] = \sum_{i=1}^{n} \left(\frac{q_i + p_i}{2}\right) \log\left(\frac{q_i + p_i}{2\sqrt{q_i p_i}}\right)$
	Relative Arithmetic - Geometric Divergence		

NON-SYMMETRIC DIVERGENCES

S. No.	Divergences	S.No.	Adjoints
1.	$\chi^2(P\|\|Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}$	1.	$\chi^2(Q\|\|P)=\sum_{i=1}^{n}\frac{(q_i-p_i)^2}{p_i}$
	Chi-square Divergence		
2.	$K(P\|\|\mathrm{Q})=\sum_{i=1}^{n}p_i\log\frac{p_i}{q_i}$	2.	$K(\mathrm{Q}\|\|\mathrm{P})=\sum_{i=1}^{n}q_i\log\frac{q_i}{p_i}$
	Relative Kullback-Leibler		
3.	$F(P\|\|Q)=\sum_{i=1}^{n}p_i\log\frac{2p_i}{p_i+q_i}$	3.	$F(Q\|\|P)=\sum_{i=1}^{n}q_i\log\frac{2q_i}{q_i+p_i}$
	Relative Jensen-Shannon		
4.	$G(P\|\|Q)=\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2p_i}\right)$	4.	$G(\mathrm{Q}\|\|\mathrm{P})=\sum_{i=1}^{n}(\mathrm{q}_i-p_i)\log\left(\frac{q_i+p_i}{2q_i}\right)$
	RelativeArithmetic-Geometric Divergences		
5.	$D(P\|\|Q)=\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)$	5.	$D(Q\|\|P)=\sum_{i=1}^{n}(q_i-p_i)\log\left(\frac{q_i+p_i}{2p_i}\right)$
	Relative J-Divergences		

Ruchi Nager and R.P.Singh [9] recently extended the symmetric and non-symmetric divergences into weighted symmetric and non-symmetric divergences and established the bounds for the same. Some new divergences are to considered in this communication in the next section and some inequalities have been established in section 3. Their concavity property has been discussed exploiting the calculus.

SECTION 2

SOME NEW WEIGHTED NON-LOGARITHMIC SYMMETRIC DIVERGENCE MEASURES

Now considering the weighted distribution corresponding to the probability distribution along with the information scheme (1.1), we define the following new non-logarithmic symmetric weighted divergences viz;

$$K_{01}^{*}(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^4}{\sqrt{(p_iq_i)^3}} \tag{2.1}$$

$$K_{02}^{*}(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(\sqrt{p_i}-\sqrt{q_i})^4}{(p_i+q_i)} \tag{2.2}$$

$$K_{03}^{*}(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(\sqrt{p_i}-\sqrt{q_i})^4}{\sqrt{p_iq_i}} \tag{2.3}$$

$$K_{04}^{*}(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^2(\sqrt{p_i}-\sqrt{q_i})^2}{(p_i+q_i)\sqrt{p_iq_i}} \tag{2.4}$$

$$K_{05}^{*}(P\|Q)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^2(\sqrt{p_i}-\sqrt{q_i})^2}{\sqrt{p_iq_i}} \tag{2.5}$$

$$K_{06}^{*}(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^4}{p_iq_i(p_i+q_i)} \tag{2.6}$$

And $$K_0(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^2}{\sqrt{p_iq_i}} \tag{2.7}$$

- **WEIGHTED EXPONENTIAL DIVERGENCE OF ORDER t AND CSISZAR'S f-DIVERGENCE**

In the same fashion, the weighted exponential divergence of order t, can be defined as:

$$K_t(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^{2(t+1)}}{(p_iq_i)\frac{2t+1}{2}} \tag{2.8}$$

where t = 0, 1, 2, 3, ...

connecting $K_t(P\|Q;W)$ with a new exponential weighted divergence, we consider the following:

$$E_K(P\|Q;W)=\frac{1}{\lfloor 0}K_0(P\|Q;W)+\frac{1}{\lfloor 1}K_1(P\|Q;W)+\frac{1}{\lfloor 2}K_2(P\|Q;W)$$
$$+\frac{1}{\lfloor 3}K_1(P\|;W)+\frac{1}{\lfloor 4}K_2(P\|Q;W)+\ldots\ldots\ldots\ldots \tag{2.9}$$

We can represent (2.9) as follows:

$$E_K(P\|Q;W)=\sum_{i=1}^{n}\frac{w_i(p_i-q_i)^2}{\sqrt{p_iq_i}}\exp\left\{\frac{(p_i-q_i)^2}{p_iq_i}\right\} \tag{2.10}$$

where $(P,Q)\in T_n\times T_n,\qquad w_i>0,\forall\, i=1,2,\ldots\ldots n$

$$\Gamma_n = [\mathrm{P} = (p_1, p_2, p_3, ..., p_n), p_i > 0, \sum_{i=1}^{n} p_i = 1], n \geq 2$$

- **CSISZAR'S f-DIVERGENCE:**

Definition:-If the function f: [0,∞) $\rightarrow R$ is convex and normalized ie f (1) = 0, then the divergence $C_f(P \| Q)$ is defined as

$$C_f(P \| Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \quad (2.11)$$

- **WEIGHTED f-DIVERGENCE:** Let f-divergence defined by Csiszar[2] equation (2.11) be given, then the weighted f-divergence is defined as

$$C_f(\mathrm{P} \| \mathrm{Q}; \mathrm{W}) = \sum_{i=1}^{n} w_i q_i f\left(\frac{p_i}{q_i}\right) \quad (2.12)$$

SECTION 3

SOME BASIC DIVERGENCE-DIFFERENCES FOR NEW WEIGHTED DIVERGENCES,THEIR FUNCTIONAL FORMS AND DERIVATIVES:

[1] $$D_{h\Delta}(P \| Q; W) = h(P \| Q; W) - \frac{1}{4}\Delta(P \| Q; W) \quad (3.1)$$

$$= \sum_{i=1}^{n} w_i q_i f_{h\Delta}\left(\frac{p_i}{q_i}\right)$$

where $$f_{h\Delta}(x) = f_h(x) - \frac{1}{4} f_\Delta(x); \qquad x > 0 \quad (3.2)$$

$$\Rightarrow f''_{h\Delta}(x) = f''_h(x) - f''_\Delta(x)$$

$$= \frac{1}{4x\sqrt{x}} - \frac{8}{(x+1)^3}$$

$$= \frac{(\sqrt{x}-1)^2[(\sqrt{x}+1)^2(x+1)+4x]}{4x^{3/2}(x+1)^3} \geq 0, \forall x > 0 \quad (3.3)$$

where $f''_h(x) = \frac{1}{4x\sqrt{x}}$, $f'_h(x) = \frac{\sqrt{x}-1}{2\sqrt{x}}$, $f''_\Delta(x) = \frac{8}{(x+1)^3}$, $f'_\Delta(x) = \frac{(x-1)(x+3)}{(x+1)^2}$

[2] $$D_{K_0\Delta}(P \| Q; W) = \frac{1}{8} K_0(P \| Q; W) - \frac{1}{4}\Delta(P \| Q; W) \quad (3.4)$$

$$= \sum_{i=1}^{n} w_i q_i f_{K_0\Delta}\left(\frac{p_i}{q_i}\right)$$

where $$f_{K_0\Delta}(x)=\frac{1}{8}f_{K_0}(x)-\frac{1}{4}f_{\Delta}(x) \quad (3.5)$$

$$=\frac{1(x-1)^2}{8\sqrt{x}}-\frac{1(x-1)^2}{4(x+1)} \qquad =\frac{(x-1)^2(\sqrt{x}-1)^2}{8\sqrt{x}(x+1)}$$

$$\Rightarrow f''_{K_o\Delta}(x)=\frac{(\sqrt{x}-1)^2[3x^4+6x^{\frac{7}{2}}+20x^3+34x^{\frac{5}{2}}+34x^{\frac{3}{2}}+20x+66x^{\frac{1}{2}}+6\sqrt{x}+3]}{32x^{\frac{5}{2}}(x+1)^3}\geq 0\forall x>0 \quad (3.6)$$

[3] $$D_{K_0h}(P\|Q;W)=\frac{1}{8}K_0(P\|Q;W)-h(P\|Q;W) \quad (3.7)$$

$$=\sum_{i=1}^{n}w_iq_i\,f_{K_0h}\left(\frac{p_i}{q_i}\right)$$

where $$f_{K_0h}(x)=\frac{1}{8}f_{K_0}-f_h(x) \quad (3.8)$$

$$=\frac{1}{8}\frac{(x+1)^2}{\sqrt{x}}-\frac{1}{2}(\sqrt{x}-1)^2$$

$$=\frac{(\sqrt{x}-1)^4}{8\sqrt{x}}$$

$$\Rightarrow f''_{K_0h}(x)=\frac{3(x-1)^2}{32x^{\frac{5}{2}}}\geq 0,\forall x>0 \quad (3.9)$$

[4] $$D_{\psi\Delta}(P\|Q;W)=\frac{1}{16}\psi(P\|Q;W)-\frac{1}{4}\Delta(P\|Q;W) \quad (3.10)$$

$$=\sum_{i=1}^{n}w_iq_i\,f_{\psi\Delta}\left(\frac{p_i}{q_i}\right)$$

$$=\frac{w}{4}\left[\frac{1}{4}f_{\psi}(x)-f_{\Delta}(x)\right]$$

hence $$f_{\psi\Delta}(x)=\frac{1}{4}\left[\frac{1}{4}f_{\psi}(x)-f_{\psi}(x)\right] \quad (3.11)$$

$$\Rightarrow f''_{\psi\Delta}(x)=\frac{1}{4}\left[\frac{1}{4}f''_{\psi}(x)-f''_{\Delta}(x)\right]$$

$$=\frac{x^3+1}{8x^3}-\frac{2}{(x+1)^3}$$

$$=\frac{(x+1)^2(x^4+5x^3+12x^2+5x+1)}{8x^3(x+1)^3}\geq 0,\forall x>0. \quad (3.12)$$

[5] $$D_{\psi K_0}(P\|Q;W)=\frac{1}{8}\left[\frac{1}{2}\psi(P\|Q;W)-K_0(P\|Q;W)\right] \quad (3.13)$$

$$=\sum_{i=1}^{n}w_iq_i\,f_{\psi K_0}\left(\frac{p_i}{q_i}\right)$$

where $$f_{\psi K_0}(x) = \frac{1}{16} f_\psi(x) - \frac{1}{8} f_{K_0}(x) \tag{3.14}$$

$$= \frac{1}{16}\frac{(x-1)^2(x+1)}{2} - \frac{1}{8}\frac{(x-1)^2}{\sqrt{x}}$$

$$= \frac{(x-1)^2(\sqrt{x}-1)^2}{16x}$$

$$\Rightarrow f''_{\psi K_0} = \frac{(\sqrt{x}-1)^2(4x^2+5x^{\frac{3}{2}}+6x+5\sqrt{x}+4)}{32x^3} \geq 0, \forall x > 0 \tag{3.15}$$

where $f'_\psi(x) = \frac{(x-1)(2x^2+x+1)}{x^2}$, $f''_\psi(x) = \frac{2(x^3+1)}{x^3}$

[6] $$D_{FK_0}(P\|Q;W) = \frac{1}{8}\left[\frac{1}{2}F(P\|Q;W) - K_0(P\|Q;W)\right] \tag{3.16}$$

$$= \sum_{i=1}^{n} w_i q_i f_{FK_0}\left(\frac{p_i}{q_i}\right)$$

where $$f_{FK_0}(x) = \frac{1}{8}\left[\frac{1}{2} f_F(x) - f_{K_0}(x)\right] \tag{3.17}$$

$$= \frac{1}{32}\frac{(x-1)^2}{x^{3\backslash 2}} - \frac{1}{8}\frac{(x-1)^2}{\sqrt{x}}$$

$$= \frac{(x-1)^4}{32x^{3\backslash 2}}$$

$$\Rightarrow f''_{FK_0} = \frac{3(5x^2+6x+5)(x-1)^2}{128x^{7\backslash 2}} \geq 0, \forall x > 0 \tag{3.18}$$

3.1 WEIGHTED NEW DIFFERENCE-DIVERGENCES IN $p_i \,\&\, q_i$ AND THEIR FUNCTIONAL FORMS:

1- $$\alpha_1(P\|Q;W) = \frac{1}{2}D_{K_0\Delta}(P\|Q;W) - D_{h\Delta}(P\|Q;\mathrm{W}) \tag{3.19}$$

$$= \frac{1}{2}\sum_{i=1}^{n} w_i q_i f_{k_0\Delta}\left(\frac{p_i}{q_i}\right) - \sum_{i=1}^{n} w_i q_i f_{h\Delta}\left(\frac{p_i}{q_i}\right)$$

$$= \frac{1}{16}\sum_{i=1}^{n} w_i\left(\frac{(\sqrt{p_i}-\sqrt{q_i})}{\sqrt{p_i}q_i(p_i+q_i)}\right)^6, setting, p_i = x, q_i = 1 \,\&\, w_i = 1 ,$$

The functional form is $$= \frac{1}{16}\left(\frac{\sqrt{x}-1}{\sqrt{x}(\mathrm{x}+1)}\right)^6 . \tag{3.20}$$

2- $$\alpha_2(P\|Q;W) = \frac{1}{4}D_{\psi\Delta}(P\|Q;W) - D_{K_0 h}(P\|Q;W) \tag{3.21}$$

$$=\frac{1}{4}\sum_{i=1}^{n} w_i q_i f_{\psi\Delta}\left(\frac{p_i}{q_i}\right)-\sum_{i=1}^{n} w_i q_i f_{K_0 h}\left(\frac{p_i}{q_i}\right)$$

$$=\frac{1}{64}\sum_{i=1}^{n} w_i \frac{(\sqrt{p_i}-\sqrt{q_i})^8}{p_i q_i (p_i+q_i)},$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is

$$f_{\alpha_2}(x)=\frac{1}{64}\frac{(\sqrt{x}-1)^8}{x(x+1)}. \tag{3.22}$$

3- $$\alpha_3(P\|Q;W)=\frac{1}{4}D_{\psi\Delta}(P\|Q;W)-\frac{1}{2}D_{K_0\Delta}(P\|Q;W) \tag{3.23}$$

$$=\frac{1}{4}\sum_{i=1}^{n} w_i q_i f_{\psi\Delta}\left(\frac{p_i}{q_i}\right)-\frac{1}{2}\sum_{i=1}^{n} w_i q_i f_{K_0\Delta}\left(\frac{p_i}{q_i}\right)$$

$$=\frac{1}{64}\sum_{i=1}^{n} w_i \frac{(p_i-q_i)^2(\sqrt{p_i}-\sqrt{q_i})^4}{p_i q_i (p_i+q_i)},$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_3}(x)=\frac{(x-1)^2(\sqrt{x}-1)^4}{x(x+1)}. \tag{3.24}$$

4- $$\alpha_4(P\|Q;W)=\frac{1}{4}D_{\psi\Delta}(P\|Q;W)-D_{h\Delta}(P\|Q;W) \tag{3.25}$$

$$=\frac{1}{4}\sum_{i=1}^{n} w_i q_i f_{\psi\Delta}\left(\frac{p_i}{q_i}\right)-\frac{1}{2}\sum_{i=1}^{n} w_i q_i f_{h\Delta}\left(\frac{p_i}{q_i}\right)$$

$$=\frac{1}{64}\sum_{i=1}^{n} \frac{w_i(p_i+6\sqrt{p_i q_i}+q_i)(\sqrt{p_i}-\sqrt{q_i})^2}{p_i q_i (p_i+q_i)},$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_4}(x)=\frac{1}{64}\frac{(x+6\sqrt{x}+1)(\sqrt{x}-1)^2}{x(x+1)}. \tag{3.26}$$

5- $$\alpha_5(P\|Q;W)=\frac{1}{4}D_{\psi\Delta}(P\|Q;W)-\frac{1}{2}D_{K_0\Delta}(\mathrm{P}\|\mathrm{Q};\mathrm{W}) \tag{3.27}$$

$$=\frac{1}{4}\sum_{i=1}^{n} w_i q_i f_{\psi\Delta}\left(\frac{p_i}{q_i}\right)-\frac{1}{2}\sum_{i=1}^{n} w_i q_i f_{K_0\Delta}\left(\frac{p_i}{q_i}\right)$$

$$=\frac{1}{64}\sum_{i=1}^{n} \frac{w_i[2(\mathrm{p}_i+q_i)+(\sqrt{p_i}-\sqrt{q_i})^2(\mathrm{p}_i-q_i)^2(\sqrt{p_i}-\sqrt{q_i})^2]}{p_i q_i (p_i+q_i)}$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_5}(x)=\frac{1}{64}\frac{2(\mathrm{x}+1)+(x-1)^2(\sqrt{x}-1)^4}{x(x+1)}. \tag{3.28}$$

6- $$\alpha_6(\mathrm{P}\,\|\,\mathrm{Q};\mathrm{W}) = \frac{1}{2}D_{\psi K_0}(P\,\|\,Q;W) - D_{K_0 h}(\mathrm{P}\,\|\,\mathrm{Q};\mathrm{W})$$ (3.29)

$$= \frac{1}{2}\sum_{i=1}^{n} w_i q_i\, f_{\psi K_0}\left(\frac{p_i}{q_i}\right) - \sum_{i=1}^{n} w_i q_i\, f_{K_0 h}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_6(P\,\|\,Q;W) = \frac{1}{16}\sum_{i=1}^{n} w_i \frac{(p_i+q_i)+(\sqrt{p_i}-\sqrt{q_i})^4}{p_i q_i}$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_6}(x) = \frac{1}{16}\frac{(x+1)(\sqrt{x}-1)^4}{x}$$ (3.30)

7- $$\alpha_7(P\,\|\,Q;W) = \frac{1}{2}D_{\psi K_0}(P\,\|\,Q;W) - \frac{1}{2}D_{K_0\Delta}(P\,\|\,Q;W)$$ (3.31)

$$= \frac{1}{2}\sum_{i=1}^{n} w_i q_i\, f_{\psi K_0}\left(\frac{p_i}{q_i}\right) - \frac{1}{2}\sum_{i=1}^{n} w_i q_i\, f_{K_0\Delta}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_7(P\,\|\,Q;W) = \frac{1}{16}\sum_{i=1}^{n} w_i \frac{(p_i+q_i)^2+[(\sqrt{p_i}-\sqrt{q_i})^2+\sqrt{p_i q_i}](\sqrt{p_i}-\sqrt{q_i})^2}{p_i q_i(p_i+q_i)}$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_7}(x) = \frac{1}{16}\frac{(x-1)^2[(\sqrt{x}-1)^2+\sqrt{x}](\sqrt{x}-1)}{x(x+1)}$$ (3.32)

8- $$\alpha_8(P\,\|\,Q;W) = \frac{1}{2}D_{\psi K_0}(P\,\|\,Q;W) - \frac{1}{2}D_{h\Delta}(P\,\|\,Q;W)$$ (3.33)

$$= \frac{1}{2}\sum_{i=1}^{n} w_i q_i\, f_{\psi K_0}\left(\frac{p_i}{q_i}\right) - \frac{1}{2}\sum_{i=1}^{n} w_i q_i\, f_{h\Delta}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_8(P\,\|\,Q;W) = \frac{1}{16}\sum_{i=1}^{n} w_i \frac{[\sqrt{p_i q_i}(p_i+q_i)+(\sqrt{p_i}-\sqrt{q_i})^2(\sqrt{p_i}-\sqrt{q_i})^4}{p_i q_i(p_i+q_i)}$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_8}(x) = \frac{1}{16}\frac{[\sqrt{x}(\mathrm{x}+1)+(\sqrt{x}-1)^2](\sqrt{x}-1)^4}{x(x+1)}$$ (3.34)

9- $$\alpha_9(P\,\|\,Q;W) = \frac{1}{2}D_{FK_0}(P\,\|\,Q;W) - D_{\psi K_0}(P\,\|\,Q;W)$$ (3.35)

$$= \frac{1}{2}\sum_{i=1}^{n} w_i q_i\, f_{FK_0}\left(\frac{p_i}{q_i}\right) - \sum_{i=1}^{n} w_i q_i\, f_{\psi K_0}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_9(P\,\|\,Q;W) = \frac{1}{32}\sum_{i=1}^{n} w_i \frac{(p_i+q_i)+(p_i-q_i)^2(\sqrt{p_i}-\sqrt{q_i})^2}{(p_i q_i)\sqrt{p_i+q_i}}$$

Setting $p_i = x, q_i = 1, w_i = 1$, the functional form is given by

$$f_{\alpha_9}(x)=\frac{1}{32}\frac{(x+1)+(x-1)^2(\sqrt{x}-1)^2}{x\sqrt{x+1}} \tag{3.36}$$

10- $$\alpha_{10}(P\|Q;W)=\frac{1}{4}D_{FK_0}(P\|Q;W)-D_{\psi K_0}(P\|Q;W) \tag{3.37}$$

$$=\frac{1}{4}\sum_{i=1}^{n}w_iq_i\,f_{FK_0}\left(\frac{p_i}{q_i}\right)-\sum_{i=1}^{n}w_iq_i\,f_{\psi K_0}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_{10}(P\|Q;W)=\frac{1}{64}\sum_{i=1}^{n}\;w_i\frac{[p_i+q_i+\sqrt{p_iq_i}+(\sqrt{p_i}-\sqrt{q_i})^2](p_i+q_i)^4}{(p_i+q_i)(p_iq_i)\sqrt{p_iq_i}}$$

Setting $p_i=x,q_i=1,w_i=1$, the functional form is given by

$$f_{\alpha_{10}}(\mathrm{x})=\frac{1}{64}\frac{[x+1+\sqrt{x}+(\sqrt{x}-1)^2](x+1)^4}{(\mathrm{x}+1)\,\mathrm{x}\,\sqrt{x}} \tag{3.38}$$

11- $$\alpha_{11}(P\|Q;W)=\frac{1}{4}D_{FK_0}(P\|Q;W)-\frac{1}{4}D_{\psi\Delta}(P\|Q;W) \tag{3.39}$$

$$=\frac{1}{4}\sum_{i=1}^{n}w_iq_i\,f_{FK_0}\left(\frac{p_i}{q_i}\right)-\sum_{i=1}^{n}w_iq_i\,f_{\psi\Delta}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_{11}(P\|Q;W)=\frac{1}{64}\sum_{i=1}^{n}\;w_i\frac{[(p_i+q_i)(p_i+4\sqrt{p_iq_i}+q_i)(\sqrt{p_i}-\sqrt{q_i})^4]}{(p_i+q_i)(p_iq_i)\sqrt{p_iq_i}}$$

Setting $p_i=x,q_i=1,w_i=1$, the functional form is given by

$$f_{\alpha_{11}}(x)=\frac{1}{64}\frac{[(x+1)(\mathrm{x}+4\sqrt{x}+1)(\sqrt{x}-1)^4]}{(x+1)x\sqrt{x}} \tag{3.40}$$

12- $$\alpha_{12}(P\|\mathrm{Q};W)=\frac{1}{4}D_{FK_0}(P\|Q;W)-D_{K_0h}(P\|Q;W) \tag{3.41}$$

$$=\frac{1}{4}\sum_{i=1}^{n}w_iq_i\,f_{FK_0}\left(\frac{p_i}{q_i}\right)-\sum_{i=1}^{n}w_iq_i\,f_{K_0h}\left(\frac{p_i}{q_i}\right)$$

$$\alpha_{12}(P\|Q;W)=\frac{1}{32}\sum_{i=1}^{n}\;w_i\frac{[p_i^2+q_i^2+2\sqrt{p_iq_i}(p_i+q_i)][(\sqrt{p_i}+\sqrt{q_i})^2(\sqrt{p_i}-\sqrt{q_i})^4]}{(p_i+q_i)(p_iq_i)\sqrt{p_iq_i}}\times[p_i^3+q_i^3+4\sqrt{p_iq_i}(p_i^2+q_i^2)+7p_iq_i(p_i+q_i)]$$

Setting $p_i=x,q_i=1,w_i=1$, the functional form is given by

$$f_{\alpha_{12}}(x)=\frac{1}{32}\frac{[x^2+1+2\sqrt{x}(x+1)][(\sqrt{x}+1)^2(\sqrt{x}-1)^4][\mathrm{x}^3+1+4\sqrt{x}(x^2+1)+7x(x+1)]}{(x+1)x\sqrt{x}} \tag{3.42}$$

13- $$\alpha_{13}(P\|Q;W)=\frac{1}{4}D_{FK_0}(P\|Q;W)-D_{h\Delta}(P\|Q;W) \tag{3.43}$$

$$=\frac{1}{4}\sum_{i=1}^{n}w_iq_i\,f_{FK_0}\left(\frac{p_i}{q_i}\right)-\sum_{i=1}^{n}w_iq_i\,f_{h\Delta}\left(\frac{p_i}{q_i}\right)$$

Setting $p_i=x,q_i=1,w_i=1$, the functional form is given by

$$f_{\alpha_{13}}(x)=\frac{1}{32}\frac{[(\sqrt{x}+1)^2(\sqrt{x}-1)^4][x^3+1+4\sqrt{x}(x^2+1)+7x(x+1)]}{(x+1)x\sqrt{x}} \tag{3.44}$$

Now we are able to establish the inequalities among $\alpha_1 - \alpha_{13}$.

SECTION 4

INEQUALITIES AMONG NEW DIFFERENCE-DIVERGENCES THEIR CONCAVITY AND NON-SYMMETRY

(i) $$\alpha_5(P\|Q;W)\le\alpha_6(P\|Q;W)\le\frac{1}{3}\alpha_{10}(P\|Q;W)\le\frac{1}{2}\alpha_9(P\|Q;W) \tag{4.1}$$

Proposition 1: The inequality holds good.

$$\alpha_5(P\|Q;W)\le\alpha_6(P\|Q;W)\ , \qquad (4.1a)$$

since the weighted f-divergence is given by

$$C_f(P\|Q;W)=\sum_{i=1}^{n}w_iq_i f\left(\frac{p_i}{q_i}\right). \tag{4.2}$$

Let us define $\quad g_{\alpha_5\circ\alpha_6}(x)=\dfrac{f''_{\alpha_5}(x)}{f_{\alpha_6}(x)}, \qquad \forall x>0$

$$=\frac{(3x^5+3x^{9/2}+12x^4+10x^{7/2}+17x^3+6x^{5/2}+17x^2+10x^{3/2}+12x+3\sqrt{x}+3)}{2(x+1)^3(2x^2+x^{3/2}+\sqrt{x}+2)} \tag{4.3}$$

Where $\quad f_{\alpha_5}(x)=\dfrac{1}{2}f_{\psi K_0}(x)-\dfrac{1}{4}f_{\psi\Delta}(x)$

$$=\frac{1}{64}\left[\frac{\{2(x+1)+(\sqrt{x}-1)^2\}(x-1)^2(\sqrt{x}-1)^2}{x(x+1)}\right] \tag{4.4}$$

And $\quad f_{\alpha_6}(x)=\dfrac{1}{2}f_{\psi K_0}(x)-f_{K_0h}(x)$

$$=\frac{1}{16}\left[\frac{(x+1)(\sqrt{x}-1)^4}{x}\right] \tag{4.5}$$

Hence(4.3) implies that

$$g'_{\alpha_5\circ\alpha_6}(x)=-\frac{3(x-1)(\sqrt{x}-1)^2}{4\sqrt{x}(x+1)^4(2x^2+x^{3/2}+\sqrt{x}+2)^2}\times(x^5+6x^{9/2}+16x^4+34x^{7/2}+35x^3+40x^{5/2}+35x^2+34x^{3/2}+16x+6\sqrt{x}+1) \tag{4.6}$$

$$\Rightarrow g'_{\alpha_5\circ\alpha_6}(x)\begin{cases}>0, & 0<x<1\\ <0, & x>1\end{cases} \tag{4.7}$$

And

$$\beta_{\alpha_5\circ\alpha_6}(x)=\sup_{x\in(0,\infty)}g_{\alpha_5\circ\alpha_6}(x)$$

$$=\lim_{x\to1}g_{\alpha_5\circ\alpha_6}(x)$$

$$=1 \tag{4.8}$$

Hence, the function $g_{\alpha_5\circ\alpha_6}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$

Hence, **concave** in $(0,\infty)$ and **non-symmetric**.

applying

$$\alpha\phi_{f_2}(a,\zeta)\le\phi_{f_1}(a,\zeta)\le\beta\phi_{f_2}(a,\zeta)$$

and (4.9)

$$\alpha\le\frac{f_1''(x)}{f_2''(x)}\le\beta,\qquad \text{where}\quad f_2''(x)\ge 0 \tag{4.10}$$

Using (4.7), (4.8), (4.9) and (4.10) together with (4.2) we get the required inequality i.e. (4.1a)

Proposition 2: The inequality holds good

$$\alpha_6(P\,||\,Q;W)\le\frac{1}{3}\alpha_{10}(P\,||\,Q;W). \tag{4.11}$$

Proof: Let us define $g_{\alpha_6\circ\alpha_{10}}(x)\le\dfrac{f''_{\alpha_6}(x)}{f''_{\alpha_{10}}(x)},\forall x>0$

(4.12)

We have $f_{\alpha_6}(x)=\dfrac{1}{2}f_{\psi K_0}(x)-f_{K_{0h}}(x)$

$$=\frac{1}{16}\frac{(x+1)(\sqrt{x}-1)^4}{x}; \tag{4.13}$$

$$f_{\alpha_{10}}(x)=\frac{1}{4}f_{FK_0}(x)-\frac{1}{4}f_{\psi\Delta}(x)$$

$$=\frac{1}{64}\frac{[x+1+\sqrt{x}+(\sqrt{x}-1)^2](x-1)^4}{x^{3/2}(x+1)} \tag{4.14}$$

hence, $g_{\alpha_6\circ\alpha_{10}}(x)=\dfrac{f''_{\alpha_6}(x)}{f''_{\alpha_{10}}(x)}$

$$=\frac{8\sqrt{x}(2x^2+x^{3/2}+\sqrt{x}+2)(x+1)^3}{(\sqrt{x}+1)^2[2(\sqrt{x}-1)^2(x^4+5x^3+12x^2+5x+1)+(x+1)(13x^4+38x^3+42x^2+38x+13)]} \tag{4.15}$$

$$g'_{\alpha_6\circ\alpha_{10}}(x)=\frac{24(\sqrt{x}-1)(x+1)^3}{\sqrt{x}(\sqrt{x}+1)^3[2(\sqrt{x}-1)^2(x^4+5x^3+12x^2+5x+1)+(x+1)(13x^4+38x^3+42x^2+38x+13)]^2}$$
$$\times[5x^8+5x^{15/2}+21x^7+19x^{13/2}+52x^6+49x^{11/2}+155x^5+87x^{9/2}+174x^4+87x^{7/2}$$
$$+155x^3+49x^{5/2}+52x^2+19x^{3/2}+21x+5\sqrt{x}+5]$$

(4.16)

$$\Rightarrow g'_{\alpha_6\circ\alpha_{10}}(x)\begin{cases}>0, & when\,0<x<1\\ <0, & when\quad x>1\end{cases} \tag{4.17}$$

we have

$$\beta g_{\alpha_6 \circ \alpha_{10}}(x) = \alpha \sup_{x\in(0,1)} g_{\alpha_6 \circ \alpha_{10}}(x)$$
$$= \lim_{x\to 1} g_{\alpha_6 \circ \alpha_{10}}(x)$$
$$= \frac{1}{3} \quad . \tag{4.18}$$

From (4.17) and (4.18),we conclude that the function $g_{\alpha_6 \circ \alpha_{10}}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$. Hence **Concave** in $x \in (0,\infty)$ and **Non-Symmetric**. Applying (4.9) and (4.10) respectively i.e.

$$\alpha\phi_{f_2}(a,\zeta) \le \phi_{f_1}(a,\zeta) \le \beta\phi_{f_2}(a,\zeta)$$

$$\alpha \le \frac{f_1''(x)}{f_2''(x)} \le \beta \ , \qquad \text{where} \quad f_2''(x) \ge 0$$

using (4.17), (4.18), (4.9) and (4.10) together with (4.2) we get the required inequality (4.11) is satisfied.

Proposition 3: The inequality holds good.

$$\frac{1}{3}\alpha_{10}(P\|Q;W) \le \frac{1}{2}\alpha_9(P\|Q;W) \ . \tag{4.19}$$

Proof : Let us define

$$g_{\alpha_{10}\circ\alpha_9}(x) = \frac{f''_{\alpha_{10}}(x)}{f''_{\alpha_9}(x)}, \quad \forall x > 0 \tag{4.20}$$

where

$$f_{\alpha_9}(x) = \frac{1}{4} f_{FK_0}(x) - \frac{1}{2} f_{\psi K_0}(x)$$
$$= \frac{1}{32}\left[\frac{(x+1)(x-1)^2(\sqrt{x}-1)^2}{x^{3/2}}\right] \tag{4.21}$$

And $f_{\alpha_{10}}(x)$ is given by (3.39) $f_{\alpha_{10}}(x) = \frac{1}{64}\frac{[x+1+\sqrt{x}+(\sqrt{x}-1)^2](x-1)^4}{x^{3/2}(x+1)}$,

hence $g_{\alpha_{10}\circ\alpha_9}(x) = \dfrac{(\sqrt{x}+1)^2[2(\sqrt{x}+1)^2(x^4+5x^3+12x^2+5x+1)+(x+1)(13x^4+38x^3+42x^2+38x+13)]}{(x+1)^3[15x^3+14x^{5/2}+13x^2+12x^{3/2}+13x+14\sqrt{x}+15]}$

(4.22)

$$\Rightarrow g'_{\alpha_{10}\circ\alpha_9}(x) = \frac{6(x-1)}{\sqrt{x}(x+1)^4[15x^3+14x^{5/2}+13x^2+12x^{3/2}+13x+14\sqrt{x}+15]^2}$$
$$\times\left[\begin{matrix}15x^8+30x^{15/2}+12x^7+174x^{13/2}+346x^6+178x^{11/2}+487x^5+258x^{9/2}+622x^4\\+258x^{7/2}+487x^3+178x^{5/2}+346x^2+174x^{3/2}+121x+30\sqrt{x}+15\end{matrix}\right] \tag{4.23}$$

$$\Rightarrow g'_{\alpha_{10}\circ\alpha_9}(x)\begin{cases} >0, & when\, 0<x<1 \\ <0, & when \quad x>1 \end{cases} \tag{4.24}$$

We have $\beta g_{\alpha_{10}\circ\alpha_9}(x) = \alpha \sup_{x\in(0,1)} g_{\alpha_{10}\circ\alpha_9}(x)$

$$= \alpha \lim_{x \to 1} g_{\alpha_{10} \circ \alpha_9}(x)$$

$$= \frac{3}{2} \tag{4.25}$$

From (4.24) and (4.25) we conclude that $g_{\alpha_{10} \circ \alpha_9}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$. Hence **Concave** and **Non-Symmetric** in $x \in (0,\infty)$.

Applying

$$\alpha \phi_{f_2}(a,\zeta) \le \phi_{f_1}(a,\zeta) \le \beta \phi_{f_2}(a,\zeta)$$

$$\alpha \le \frac{f_1''(x)}{f_2''(x)} \le \beta \ , \qquad \text{where} \quad f_2''(x) \ge 0 \quad \forall a,\zeta \in (0,\infty)$$

Using (4.24), (4.25), (4.9), and (4.10) together with (4.2) we get the required inequality

$$\frac{1}{3}\alpha_{10}(P \| Q;W) \le \frac{1}{2}\alpha_9(P \| Q;W).$$

Now combining propositions 1, 2, and 3, we get the required inequality (4.1) i.e.

$$\alpha_5(P \| Q;W) \le \alpha_6(P \| Q;W) \le \frac{1}{3}\alpha_{10}(P \| Q;W) \le \frac{1}{2}\alpha_9(P \| Q;W) \ .$$

SECTION 5

SOME MORE INEQUALITIES, CONCAVITY AND NON-SYMMETRY OF NEW WEIGHTED DIFFERENCE-DIVERGENCES

(ii) The inequality $$\alpha_7(P \| Q;W) \le \frac{1}{3}\begin{cases} \alpha_{12}(P \| Q;W) & \le \frac{1}{2}\alpha_9(P \| Q;W) \\ \alpha_{13}(P \| Q;W) \end{cases} \tag{5.1}$$

The above inequality is again proved through the following propositions.

Proposition 4: The inequality $$\alpha_7(P \| Q;W) \le \frac{1}{3}\begin{cases} \alpha_{12}(P \| Q;W) \\ \alpha_{13}(P \| Q;W) \end{cases} \tag{5.2}$$

Proof: Let us define $$g_{\alpha_7 \circ \alpha_{12}}(x) = \frac{f''_{\alpha_7}(x)}{f''_{\alpha_{12}}(x)}, \quad \forall x \in (0,\infty) \tag{5.3}$$

where

$$f_{\alpha_7}(x) = \frac{1}{2} f_{\psi K_0}(x) - \frac{1}{2} f_{K_0 \Delta}(x)$$

$$= \frac{1}{16}\left[\frac{(x-1)^2\{(\sqrt{x}-1)^2 + \sqrt{x}\}(\sqrt{x}-1)}{x(x+1)}\right] \tag{5.4}$$

and

$$f_{\alpha_{12}}(x) = \frac{1}{4} f_{FK_0}(x) - \frac{1}{2} f_{K_0 h}(x) \tag{5.5}$$

$$f_{\alpha_{12}}(x)=\frac{1}{32}\left[\frac{\{(x^2+1)+2\sqrt{x}(x+1)\}\{(\sqrt{x}+1)^2(\sqrt{x}-1)^4\}\{(x^3+1)+4\sqrt{x}(x^2+1)+7x(x+1)}{x\sqrt{x}(x+1)}\right] \quad (5.6)$$

Hence

$$g_{\alpha_9\circ\alpha_{12}}(x)=\frac{2\sqrt{x}[8x^5+7x^{9/2}+30x^4+20x^{7/2}+31x^3+31x^2+3x^2(\sqrt{x}-1)^2+20x^{3/2}+30x+7\sqrt{x}+8]}{[15x^6+30x^{11/2}+72x^5+114x^{9/2}+137x^4+160x^{7/2}+96x^3+160x^{5/2}+137x^2+114x^{3/2}+72x+30\sqrt{x}+15]} \quad (5.7)$$

$$\Rightarrow g'_{\alpha_7\circ\alpha_{12}}(x)=\frac{6(x-1)(\sqrt{x}-1)^2(x+1)^2}{\sqrt{x}[15x^6+30x^{11/2}+72x^5+114x^{9/2}+137x^4+160x^{7/2}+96x^3+160x^{5/2}+137x^2+114x^{3/2}+72x+30\sqrt{x}+15]^2} \quad (5.8)$$

$$\times\begin{bmatrix}20x^7+75x^{13/2}+274x^6+634x^{11/2}+1294x^5+1685x^{9/2}+2144x^4+2124x^{7/2}+2144x^3\\+1685x^{9/2}+1274x^2+634x^{3/2}+274x+75\sqrt{x}+20\end{bmatrix}$$

$$\Rightarrow g'_{\alpha_7\circ\alpha_{12}}(x)\begin{cases}>0, & when\, 0<x<1\\<0, & when \quad x>1\end{cases} \quad (5.9)$$

and

$$\beta g_{\alpha_7\circ\alpha_{12}}(x)=\sup_{x\in(0,\infty)} g_{\alpha_7\circ\alpha_{12}}(x)$$

$$=\lim_{x\to1} g_{\alpha_7\circ\alpha_{12}}(x) \quad =\frac{1}{3} \quad (5.10)$$

From (5.9) and (5.10),we conclude that the function $g_{\alpha_6\circ\alpha_{10}}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$. Hence, **Concave** in $x\in(0,\infty)$ and **Non-Symmetric**. Applying

$$\alpha\phi_{f_2}(a,\zeta)\le\phi_{f_1}(a,\zeta)\le\beta\phi_{f_2}(a,\zeta)$$

$$\alpha\le\frac{f_1''(x)}{f_2''(x)}\le\beta\ , \qquad \text{where} \quad f_2''(x)\ge0$$

Using (5.9), (5.10), (4.9) and (4.10) together with (4.2) we get required inequality. i.e.

$$\alpha_7(P\|Q;W)\le\frac{1}{3}\alpha_{12}(P\|Q;W).$$

Proposition 5: The inequality $\alpha_7(P\|Q;W)\le\frac{1}{3}\alpha_{13}(P\|Q;W).$ (5.11)

Proof: Let us define

$$g_{\alpha_7\circ\alpha_{12}}(x)=\frac{f''_{\alpha_7}(x)}{f''_{\alpha_{12}}(x)},\quad \forall x\in(0,\infty) \quad (5.12)$$

where $f_{\alpha_7}(x)$ is given by (4.4) and $f_{\alpha_{13}}(x)$ is given as follows:

$$f_{\alpha_{13}}(x)=\frac{1}{32}\left[\frac{\{(\sqrt{x}+1)^2(\sqrt{x}-1)^4\}\{(x^3+1)+4\sqrt{x}(x^2+1)+7x(x+1)\}}{x\sqrt{x}(x+1)}\right] \quad (5.13)$$

hence, we have

$$g_{\alpha_7\circ\alpha_{13}}(x)=\frac{2\sqrt{x}[8x^5+7x^{9/2}+30x^4+20x^{7/2}+31x^3+31x^2+3x^2(\sqrt{x}-1)^2+20x^{3/2}+30x+7\sqrt{x}+8]}{[15x^6+30x^{11/2}+78x^5+126x^{9/2}+145x^4+164x^{7/2}+36x^3+164x^{5/2}+145x^2+126x^{3/2}+78x+30\sqrt{x}+15]}$$

$\forall x>0$ (5.14)

$$\Rightarrow g'_{\alpha_7\circ\alpha_{13}}(x)=\frac{6(x-1)(\sqrt{x}-1)^2(x+1)^2}{\sqrt{x}[15x^6+30x^{11/2}+78x^5+126x^{9/2}+145x^4+164x^{7/2}+36x^3+164x^{5/2}+145x^2+126x^{3/2}+78x+30\sqrt{x}+15]^2} \quad (5.15)$$

$$\times\begin{bmatrix}20x^8+35x^{15/2}+136x^7+129x^{13/2}+272x^6-197x^{11/2}+536x^5-223x^{9/2}+504x^4\\-223x^{7/2}+536x^3-197x^{5/2}+272x^2+129x^{3/2}+136x+35\sqrt{x}+20\end{bmatrix}$$

$$\Rightarrow g'_{\alpha_7\circ\alpha_{13}}(x)\begin{cases}>0, & when\, 0<x<1\\<0, & when \quad x>1\end{cases} \quad (5.16)$$

and

$$\beta g_{\alpha_7\circ\alpha_{13}}(x)=\sup_{x\in(0,\infty)} g_{\alpha_7\circ\alpha_{13}}(x)$$

$$=\lim_{x\to 1} g_{\alpha_7\circ\alpha_{13}}(x)$$

$$=\frac{1}{3} \quad (5.17)$$

Hence we conclude from (5.16) and (5.17), that the function $g_{\alpha_6\circ\alpha_{10}}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$. Hence, **<u>Concave</u>** in $x\in(0,\infty)$ and **<u>Non-Symmetric</u>**. Applying

$$\alpha\phi_{f_2}(a,\zeta)\le\phi_{f_1}(a,\zeta)\le\beta\phi_{f_2}(a,\zeta)$$

$$\alpha\le\frac{f_1''(x)}{f_2''(x)}\le\beta\ , \qquad \text{where} \quad f_2''(x)\ge 0$$

Using (5.16), (5.17), (4.1a) and (4.10) together with (4.2),

we get the required inequality (5.11) ie

$$\alpha_7(P\|Q;W)\le\frac{1}{3}\alpha_{13}(P\|Q;W).$$

Now combining proposition 4 and 5,we get the required inequality. i.e. (5.2)

$$\alpha_7(P\|Q;W)\le\frac{1}{3}\begin{cases}\alpha_{12}(P\|Q;W)\\\alpha_{13}(P\|Q;W)\end{cases}$$

<u>Proposition 6</u>: The inequality $\frac{1}{3}\alpha_{12}(P\|Q;W)\le\frac{1}{2}\alpha_9(P\|Q;W).$ (5.18)

Proof: Let us define $g_{\alpha_{12}\circ\alpha_9}(x)=\frac{f''_{\alpha_{12}}(x)}{f''_{\alpha_9}(x)},\quad\forall x>0$ (5.19)

where $f_{\alpha_9}(x)$ is given by (3.37) and $f_{\alpha_{12}}$ are given by (3.43) respectively.

Hence $g_{\alpha_{12}\circ\alpha_9}(x)=\frac{f''_{\alpha_{12}}(x)}{f''_{\alpha_9}(x)},\quad\forall x>0$

$$=\frac{2\sqrt{x}\left[15x^{6}+30x^{11/2}+72x^{5}+114x^{9/2}+137x^{4}+160x^{7/2}+96x^{3}+160x^{5/2}+137x^{2}+114x^{3/2}+72x+30\sqrt{x}+15\right]}{(x+1)^{3}\left[15x^{3}+14x^{5/2}+13x^{2}+12x^{3/2}+13x+14\sqrt{x}+15\right]}$$

(5.20)

$$\Rightarrow g'_{\alpha_{12}\circ\alpha_{9}}(x)=\frac{6(x-1)(\sqrt{x}-1)^{2}}{\sqrt{x}(x+1)^{4}[15x^{3}+14x^{5/2}+13x^{2}+12x^{3/2}+13x+14\sqrt{x}+15]^{2}} \tag{5.21}$$

$$\times\begin{bmatrix}20x^{7}+75x^{13/2}+274x^{6}+634x^{11/2}+1274x^{5}+1685x^{9/2}+2144x^{4}+2124x^{7/2}+2144x^{3}\\+1685x^{9/2}+1274x^{2}+634x^{3/2}+274x+75\sqrt{x}+20\end{bmatrix}$$

$$\Rightarrow g'_{\alpha_{12}\circ\alpha_{9}}(x)\begin{cases}>0, & when\, 0<x<1\\ <0, & when\quad x>1\end{cases} \tag{5.22}$$

and
$$\beta g_{\alpha_{12}\circ\alpha_{9}}(x)=\sup_{x\in(0,\infty)} g_{\alpha_{12}\circ\alpha_{9}}(x)$$
$$=\lim_{x\to 1} g_{\alpha_{12}\circ\alpha_{9}}(x)$$
$$=\frac{3}{2} \tag{5.23}$$

Hence we conclude that the function $g_{\alpha_{12}\circ\alpha_{9}}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$. Hence, $g_{\alpha_{12}\circ\alpha_{9}}(x)$ **Concave** in $x\in(0,\infty)$ and **Non-Symmetric** about x=1.

Now applying

$$\alpha\phi_{f_2}(a,\zeta)\le\phi_{f_1}(a,\zeta)\le\beta\phi_{f_2}(a,\zeta)$$

$$\alpha\le\frac{f_1''(x)}{f_2''(x)}\le\beta\ , \qquad\qquad \text{where}\quad f_2''(x)\ge 0$$

Using (5.22), (5.23), (4.9) and (4.0) together with (4.2) we get the required inequality (5.18) ie

$$\frac{1}{3}\alpha_{12}(P\|Q;W)\le\frac{1}{2}\alpha_{9}(P\|Q;W).$$

Now combining proposition 4 and 5,and 6 we get the required inequality (5.1).

$$\alpha_{7}(P\|Q;W)\le\frac{1}{3}\begin{cases}\alpha_{12}(P\|Q;W) & \le\frac{1}{2}\alpha_{9}(P\|Q;W)\\ \alpha_{13}(P\|Q;W)\end{cases}$$

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STOCHASTIC MODEL TO ESTIMATE THE INSULIN SECRETION USING NORMAL DISTRIBUTION

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ABSTRACT :

Glucagon Like Peptide 1 (GLP-1) exerts beneficial antidiabetic actions via effects on pancreatic β and α cells. Previous studies have focused on the improvements in β cell function, while the inhibition of α cell secretion has received less attention. The aim of this research was to quantify the relative contribution of GLP-1 induced insulin increase and glucagon decrease, on the overall glucose lowering effect of native GLP-1 infusions in a clinical setting with the help of Painleve analysis. In some cases we find that it has only the conditional Painleve property and in other cases, just the painleve property. We also obtained special solutions of Painleve analysis. In that, one of the solution (i.e) the reduction of nonlinear diffusion equation to Riccati equation was used for the insulin secretion.

Keywords: ***Insulin, GLP-1, Painleve Property, Riccati Equation & Normal Distribution.***

1. INTRODUCTION:

The incretin hormone Glucagon Like Peptide 1 (GLP-1), as well as GLP-1 analogues that are now being used for the treatment of patients with type 2 diabetes, potently suppresses α cell secretion [3] & [4]. Despite their hyperglycemia, patients with type 2 diabetes tend to have elevated fasting glucagon levels and exaggerated glucagon responses to meal ingestion [5]. Since the hyperglucagonemia is thought to contribute to the hyperglycemia of these patients by increasing hepatic glucose production (HGP) [6], it follows that the glucagonostatic effect of GLP-1 may be as important clinically as its insulinotropic effect [6]. Glucose induced inhibition of α cell secretion may be impaired in type 2 diabetic patients, but pharmacological amounts of GLP-1 have been shown to restore α cell sensitivity to glucose [5] & [6], and we recently demonstrated that the glucagon suppressive effect of GLP-1 was similar in patients with type 2 diabetes and healthy control subjects.

Thus GLP-1 potently influences both β and α cell secretion in patients with type 2 diabetes, but the relative roles of these two effects in relation to the overall glucoselowering action of GLP-1 are unclear. In the present studies, we sought to determine the importance of the glucagonostatic effect by measuring its contribution to changes in glucose turnover induced by the infusion of GLP-1 in pharmacological doses. We employed the pancreatic clamp technique [3] using somatostatin to block the endogenous secretion from the islets while substituting insulin and/or glucagon levels by infusions designed to mimic either basal levels and/or responses to a GLP-1 infusion rate known to normalize blood glucose in patients with type 2 diabetes [4]. All examinations were done in the fasting state with plasma glucose (PG) clamped at individual fasting PG (FPG) levels. The amount of glucose infused to maintain the clamp was expected to accurately reflect the influence of the endocrine perturbations on glucose turnover.

In recent years, much attention has been focused on higher order non linear partial differential equations, known as evolution equations. Such nonlinear equations often occur in the description of chemical and biological phenomena. Their analytical study has been drawing immense interest. In [1], a nonlinear partial differential equation is integrable if all its exact reductions to ordinary differential equations have the Painleve property: that is, to have no movable singularities other than poles. This approach poses an obvious operational difficulty in finding all exact reductions. The reduction of $u_t = \mu u^2 u_x + Duu_{xx} + Du_x^2$ to Riccati equation $f(z) = \sqrt{3c}((-A\sin X + B\cos X)/(A\cos X + B\sin X))$ was used to find the insulin secretion.

2. NOTATIONS:

$GLP-1$	-	Glucagon like Peptide 1
HGP	-	Hepatic Glucose Production
β	-	Intensity
φ	-	Arbitrary Function
r_j	-	Resonances
A	-	Shape Parameter
B	-	Scale Parameter

3. PAINLEVE ANALYSIS:

In [1], a nonlinear partial differential equation is integrable if all its exact reductions to ordinary differential equations have the Painleve property: that is, to have no movable singularities other than poles. This approach poses an obvious operational difficulty in finding all exact reductions. This difficulty was circumvented by [12] by postulating that a partial differential equation has the Painleve property if its solutions are single - valued about a movable singular manifold

$$\varphi(z_1, z_2, \dots, z_n) = 0$$

where φ is an arbitrary function. In other words, a solution $u(z_i)$ of a partial differential equation should have a Laurent - like expansion about the movable singular manifold $\varphi = 0$:

$$u(z_i) = [\varphi(z_i)]^{\alpha} \sum_{j=0}^{\infty} u_j(z_i)\varphi(z_i)^j \qquad (1)$$

Where α a is a negative integer. The number of arbitrary functions in expansion (1) should be equal to the order of the partial differential equation. Inserting expansion (1) into the targeted equation yields a recurrence formula that determines $u_n(z_i)$ for all $n > 0$, except for a finite number of $r_1, r_2, \dots, r_j > 0$, called resonances. For some equations, the recurrence formulas at the resonance values may result in constraint equations for the movable singular manifold which implies that it is no longer completely arbitrary. In such cases, one can say that the equation has the Conditional Painleve Property [7]. The Painleve property is a sufficient condition for the integrability or solvability of equations. Meanwhile, various authors have applied this approach to other nonlinear partial differential equations to decide whether or not these equations are integrable. Recent investigations of [2] regarding the Painleve analysis also yield a systematic procedure for obtaining special solutions when an equation possesses only the conditional Painleve property. From [2] & [7] proposed the nonlinear diffusion equation

$$u_t = Du_{xx} + \beta u(1-u) \qquad (2)$$

as a model for the propagation of a mutant gene with an advantageous selection of intensity β. From [7] has considered the extended form of equation (2) as

$$u_t = \beta u^p(1-u^q) + D(u^m u_x)_x \qquad (3)$$

For Painleve analysis and obtained special solutions for various cases of p, q and m.

In this paper we consider

$$u_t = \beta u^p(1-u^q) + \mu u^s u_x + D(u^m u_x)_x \qquad (4)$$

This is a generalization of (3) for the Painleve analysis. This equation has several interesting limiting cases which have already been studied:

(i) When $\mu = m = 0, p = 1$and $q \neq 0$, equation (4) is reduced to the generalized Fisher equation. For $q = 1$, equation (4) reduces to the Fisher equation and for $q = 2$, (4) reduces to the Newell Whitehead equation.

(ii) If we take $\beta = m = 0$, then equation (4) is reduced to the generalized Burgers equation. With $s = 1$ and $\beta = m = 0$, equation (4) gives the Burgers equation, which describes the far field of wave propagation in nonlinear dissipative systems [13].

(iii) When $m = 0, p = 1$ and $q = s$, equation (4) is reduced to the generalized Burger - Fisher equation [11].

The behavior of solutions of equation (4) at a movable singular manifold, $\varphi(x,t) = 0$ is determined by a leading order analysis where by one makes the substitution

$$u(x,t) = u_0(x,t)[\varphi(x,t)]^{\alpha} \quad (5)$$

and balances the most singular or dominant terms. Substituting (5) into (4), we obtain three possible values for α as follows:

Case (i):

$p + q > m \geq s$: Balancing the dominant terms $u^{p+q}, mu^{m-1}u_x^2$ and $u^m u_{xx}$, we obtain

$$\alpha = -2/(p + q - m - 1) \quad (6)$$

and $\beta u_0^{p+q-m-1} = 2D(p + q + m + 1)\varphi_x^2/(p + q - m - 1)^2$

Case (ii):

$p + q > s > m = 0$: Balancing the dominant terms u^{p+q} and $u^s u_x$, we obtain

$$\alpha = -1/p + q - s - 1$$

and $\beta u_0^{p+q-s-1} = (-\mu/p + q - s - 1)\varphi_x$

For $p + q = -m + 2s + 1 > s > m$,

$$\alpha = -1/(s - m) \quad (7)$$

Here we have two branches for u_0 as follows:

Branch (i): $u_0 = -(k + 1)\varphi_x$ & Branch (ii): $u_0 = k\varphi_x$ where $k = 1, 2, 3, \ldots$

$$m = ((2k + 1)^2 - 9)/4 \text{ and } \beta = \mu = D = 1 \quad (8)$$

Case (iii):

$s > m \geq p + q$: Balancing the dominant terms $u^s u_x, mu^{m-1}u_x^2$ and $u^m u_{xx}$, we obtain $\alpha = -1/(s - m)$

and $$u_0^{s-m} = (D/\mu)((1 + s)/(s - m))\varphi_x \quad (9)$$

We have the following lemma as a result.

Lemma:

For all combinations of integer values of p, q, m and s, the leading order singularity of equation (4) is

(i) A movable pole for all combinations with $(p + q - m - 1)$ is equal to 1 or 2 for case (i), with $(p + q - s - 1)$ being equal to 1 for case (ii), and with $s - m$ being equal to 1 for case (iii).

(ii) A rational branch point for all combinations with $(p + q - m - 1) > 2$ for case (i). $(p + q - s - 1) > 1$ for case (ii) and $s - m > 1$ for case (iii).

The powers of φ, at which the arbitrary coefficient appears in the series, that is, the resonances are determined by setting

$$u(x,t) = u_0(x,t)\big(\varphi(x,t)\big)^{\alpha} + p\big(\varphi(x,t)\big)^{\alpha+r}$$

and balancing the most singular terms of equation (4) again. We obtain for case (i), using the value of α given by (6),

$$p\{(r + \alpha)^2 + (2m\alpha - 1)(r + \alpha)[m\alpha(\alpha - 1) + m(m - 1)\alpha^2 - 2(p + q)(p + q + m + 1)/(p + q - m - 1)^2]\} = 0$$

with solutions

$$r = -1, \quad 2[1 - \alpha(m + 1)]$$

However, for case (ii), with value α given in (7) and for a particular value of m given by (8), we obtain for branch (i)

$$p\{(2m\alpha(r+\alpha)+\alpha^2m(m-1)+(r+\alpha)(r+\alpha-1)+\alpha(\alpha-1)m)-(s\alpha+(r+\alpha))(k+1)—(-m+2s+1)(k+1)^2\}=0$$

with solutions $r=-1,\ (k+3)-2\alpha(m+1)\}$

and for branch (ii)

$$p\{(2m\alpha(r+\alpha)+\alpha^2m(m-1)+(r+\alpha)(r+\alpha-1)+\alpha(\alpha-1)m)+(s\alpha+(r+\alpha))k+(—m+2s+1)k^2\}=0$$

with solutions $r=-1\ \&\ (2-k)-2\alpha(m+1)$

For case (iii), we get

$$p\{r^2+r(2\alpha(m+1))-r+\alpha^2(1+m)^2-\alpha(1+m)+\alpha s(r+\alpha)u_0^{s-m}\}=0$$

with solutions $r=-1\ \&\ -\alpha(1+s)$ (10)

By using the above lemma, we consider the following cases;

(i) $m=0, s=0, p=1, q=2$

(ii) $m=0, s=1, p=1, q=2$

(iii) $m=0, s=1, p=0, q=0$

(iv) $m=1, s=2, p=0, q=0$

(v) $m=2, s=3, p=q=1,$

In which equation (4) has a movable pole as leading order singularity, and therefore, it may have a valid Laurent Expansion.

Now consider the case (iv): Equation (4) with $m=1, s=2, p=q=0$

In this case, equation (4) becomes $u_t=\mu u^2u_x+Duu_x+Du_x^2$ (11)

Using (9) and (10), we obtain $u_0=(3D/\mu)\varphi_x$

and the resonances are $r=-1$ and 3. Hence, we take the Laurent expansion of the form

$$u=u_0\varphi^{-1}+u_1+u_2\varphi+u_3\varphi^2 \tag{12}$$

Substituting (12) into (11) and collecting coefficients of equal powers of φ, we have

φ^{-4}: $u_0=(3D/\mu)$

φ^{-3}: $u_1=0$

φ^{-2}: $u_2=(-1/3D)\sigma_t$

φ^{-1}: $0Xu_3=0$ (13)

Equation (13) shows that u_3 is an arbitrary function. Therefore (11) possesses the Painleve property.

Reduction of $u_t=\mu u^2u_x+Duu_x+Du_x^2$ to Riccati Equation:

Let $u=f(z)$, where $z=x-ct$ (14)

Substituting (14) into (11) with $\mu=D=1$ we obtain

$$cf'+f^2f'+ff''+f'^2=0 \tag{15}$$

Integrating (15) once, we get $f'=-(c+(f^2/3))$ (16)

Equation (16) is a Riccati Equation, which can be linearized through the transformation

$$f=3y'/y \tag{17}$$

Substituting (17) into (16), we get $y''=(-c/3)y(z)$ (18)

which is a second order linear differential equation. Solving (18), we obtain

$$y(z)=A\cos X+B\sin X \tag{19}$$

Using (19) then (17) becomes

$$f(z)=\sqrt{3c}((-A\sin X+B\cos X)/(A\cos X+B\sin X)) \tag{20}$$

4. EXAMPLE:

Ten patients with T2DM had their FPG clamped during five different occasions. On 1 day, GLP-1 was infused in pharmacologically doses to stimulate endogenous insulin and suppress endogenous glucagon secretion

[5] & [6]. The resulting glucose infusion rates required to maintain PG levels were viewed as a measure of the overall effect of GLP-1 on glucose turnover. This requirement was compared to pancreatic endocrine clamps using somatostatin to block endogenous insulin and glucagon secretion, while basal or stimulated insulin was replaced, and on 2 days intrahepatic glucagon concentrations were mimicked in order to isolate beta and alpha cell effects of GLP-1 [3] & [4]. When mimicking GLP-1 induced insulin secretion without glucagon suppression and glucagon suppression without elevated plasma insulin, the glucose requirements were found to be similar and half of what was required on the day when GLP-1 was infused. Infusions mimicking GLP-1 induced stimulation of insulin and inhibition of glucagon resulted in a glucose demand comparable to the GLP-1 infusion alone. Thus, this was interpreted to reflect the overall endocrine function on glucose turnover; that is, decreased HGP and increased peripheral disposal. Based on these findings it was suggested that GLP-1 induced insulin stimulation and glucagon inhibition contribute equally to the glucose lowering effect of GLP-1 in T2DM patients [8-10].

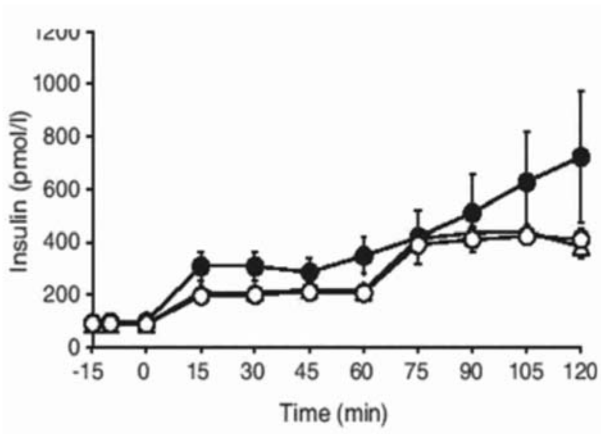

Figure (1): Insulin secretion stimulated by GLP-1 (Day 1) or infused to copy the GLP-1 elicited insulin curves (Day 4 and 5)

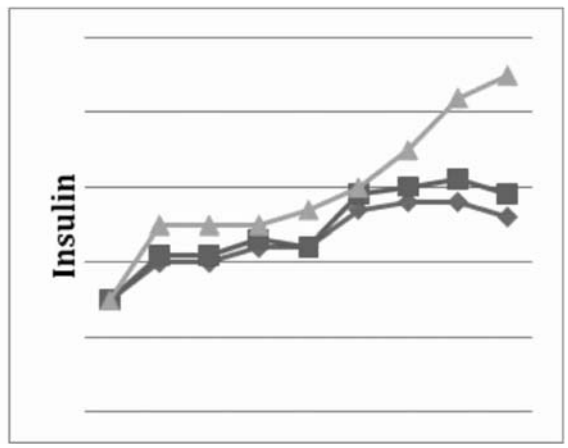

Figure (2): Insulin secretion stimulated by GLP-1 (Day 1) or infused to copy the GLP-1 elicited insulin curves (Day 4 and 5) using normal distribution

5. CONCLUSION:

Painleve analysis with normal distribution gives the same as the medical report. There is no significance difference between medical and mathematical reports. The medical reports are beautifully fitted with the mathematical model. Hence the mathematical report {Figure (2)} is coincide with the medical report {Figure (1)}.

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REVIEW ON ESTIMATION OF CORRELATION BETWEEN VARIOUS PIXEL INTENSITIES OF MICROARRAY SPOTS

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ABSTRACT :
In cDNA microarray experiments, the measurement of interest is signal intensity ratio of spots. Each spot have four types of pixel intensities namely red foreground, green foreground, red background and green background. The uncertainty associated with the signal intensity ratio depends on the correlation between pixel intensities of spots. It is important to estimate these correlations of pixel intensities of spots to propagate the uncertainty associated with the intensity ratio as well as to address the systematic errors arise from the chip artifacts. The present study is a review to identify various methods to estimate the correlation of various pixel intensities of microarray spots. Articles published from 1994 to 2015 are searched in various databases like PubMed, Scopus, Web of Science, and Google Scholar. Six articles are selected for the review based on selection criteria. We observed that there is very little literature available which deals with the impotence of the correlation between various pixel intensities of microarray spots.

Keywords: ***cDNA microarray, Pixel intensities, Intensity ratio, Correlation, Pixel correlations***

I INTRODUCTION

In cDNA microarray experiments, the measurement of interest is average of red to green pixel intensity ratio at each spot that gives the expression level of a particular gene in a diseased sample compared to normal. The cDNA microarray image files have thousands of spots and each spot is differentiated into foreground area and background area [1]. Each pixel of these areas has both red and green pixel intensity measurements (Fig: 1). For a spot, let $\overline{R}_f$, $\overline{R}_b$, $\overline{G}_f$ and $\overline{G}_b$ denote the average foreground red pixel intensity, average background red pixel intensity, average foreground green pixel intensity and average background green pixel intensity respectively. The background corrected intensity ratio for a spot is given by $y = \frac{\overline{R}_f - \overline{R}_b}{\overline{G}_f - \overline{G}_b}$ [2,3]. Due to the presence of chip artefacts, there will be correlation between various pixel intensities [2] (i.e. between red foreground (R_f) and red background (R_b) ; between green foreground (G_f) and green background (G_b) ;between red foreground (R_f) and green background (G_b) ;between green foreground (G_f) and red background (R_b) ; between red foreground (R_f) and green foreground (G_f);between red background (R_b) and green background (G_f) within a spot.

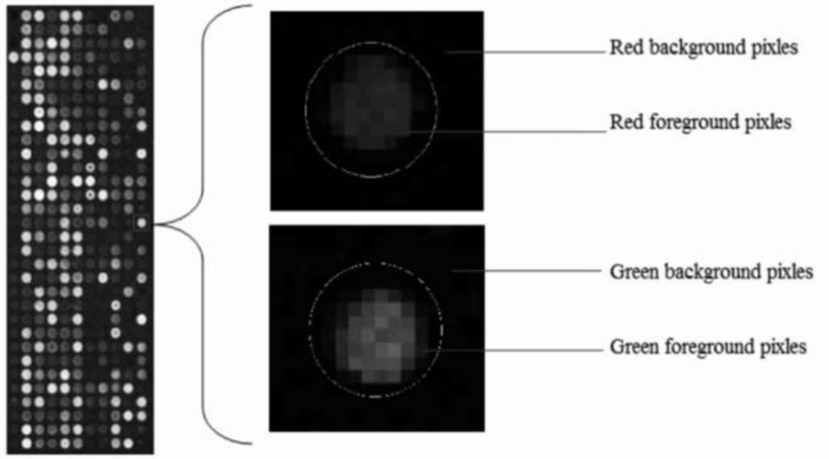

Figure 1: Image of a single spot from a cDNA microarray image file

The estimated value of any measurement may deviate from the true value and this deviation is called error or uncertainty associated with that estimate. Binu et.al (2012) demonstrated that the uncertainty in the estimate of intensity ratio depends on the correlation between various pixel intensities [4]. The correlation between intensities of green and red foreground pixels in a spot can be calculated using Pearson correlation coefficient, because we have paired values of intensities for red and green foreground pixels. Similarly, we can estimate the correlation between intensities of red and green background pixels using Pearson correlation coefficient. However, it is not straight forward to estimate the remaining four correlations (between red foreground and red background; between green foreground and green background; between red foreground and green background; between green foreground and red background) by means of Pearson's correlation coefficient because the number of pixels in the foreground and in the background differs and the position also differs. Hence this review is trying to identify the methodology to estimate the above mentioned four correlations of pixel intensities of microarray spots.

II METHODOLOGY

Articles published from 1st January 1994 to 1st June 2015 are searched in various databases Pubmed, Scopus, Web of Science and Google Scholar. The key words used for the review includes, "correlation of pixels and microarray", "relationship between pixels and microarray". Each keyword is entered in the above databases and articles are selected based on the following criteria.

First the titles of the articles published since 1994 were scrutinized. The titles which are found to be relevant for the study were selected and abstract of these articles are studied. Full text of those abstracts which appeared to be relevant for the review was obtained. Finally the full articles which found to be relevant for the current study were selected.

III RESULTS AND DISCUSSION

The results of initial screening and number of articles selected for the study are given in the table 1.

Table 1: Details of article search in the various databases.

Database	Number scrutinized			Selected for review
	Title	Abstract	Full text	
PubMed	7	3	3	3
Scopus	21	4	3	3
Web of Science	12	10	6	3
Google Scholar	11	2	2	2

We selected six articles in this review, after eliminating the duplicated files obtained from various databases. More information about the selected articles are given below.

In 2001, Carl et al., in the article entitled "Image metrics in the statistical analysis of DNA microarray data" used pixel-by-pixel analysis of individual spots to estimate the sources of error and establish the precision and accuracy with which gene expression ratios are determined. According to the authors the average correlation coefficient *r* is a good indicator of overall scan quality, and the authors estimated the correlation by pairing the red and green intensities of the pixels of the spots. The authors observed that high correlation between red and green channels appears to arise from an intrinsic granularity generated during array fabrication and hybridization. In some spots however, red and green signals fluctuate independently of each other causing the apparent gene expression ratio to vary from one place in the spot to the next [5].

In 2003, Qian et al., estimated the correlation between pixel intensities of spots in microarray chip by means of Pearson product moment correlation coefficient. According to the authors the average correlation coefficient should be independent of the chip distance if there are no chip artifacts. They observed that the closer two genes are on the chip, the higher their average correlation coefficient is, which is an indication for chip artifacts [6]. In 2005, Claus et al., compared five spatial correlation structures like exponential, Gaussian, linear, rational quadratic and spherical for selected genes that accommodate errors by allowing the spatial correlation among pixels in the microarray slide. In this study the authors considered only the immediate neighboring pixels in the spatial correlation models and not mentioned the method to estimate the correlations of various pixel intensities of a spot [7].

In 2006, R Nagarajan et al., used correlation statistics for segmentation of spots in microarray. In this article the authors used Pearson's correlation coefficient to find the correlation between red and green pixels of the spot. Also, they used correlation statistics like Pearson's correlation and Spearman rank correlation to segment pixels belonging to the foreground and background by statistically comparing only the adjacent rows and columns of pixels in the spot. The performance of correlation-based segmentation is compared to clustering-based (PAM, k-means) and seeded region growing techniques (SPOT). It is shown that correlation-based segmentation is useful in flagging poorly hybridized spots, thus minimizing false-positives. In this article the authors are not provided any method to estimate the correlation between foreground and background pixels of the same spot instead they provided a segmentation method which is based on the correlation between pixels of the adjacent rows or columns of the pixels in the spot grid [8].

In 2010, Bergmann et al., proposed two quantities as measure of spot quality which uses the spatial correlation between intensities of pixels in a spot. The spatial structure assumes that correlation decays with increasing distance. Newton-Raphson algorithm is used to calculate the optimal estimate for correlation between pixels. The main limitations of proposed methods are the assumption of multivariate normal distribution of intensities of pixels in the spot which may not be always true and considered only correlation between neighboring pixels [9].

Binu et al., identified the role of correlation between pixel intensities within a spot in cDNA microarray chip on uncertainty estimation. The authors demonstrated the importance of correlation between the intensities of pixels while estimating the uncertainty associated with each spot. They observed that as correlation between pixels in the spot increases the uncertainty associated with the intensity ratio decreases. However in that study the author's assumed equal correlation between the various types of pixel intensities of spot which is not true [2,4].

The estimate of correlations will help us to propagate the error associated with the functions of genes which can be incorporated in statistical analysis like cluster analysis, multivariate analysis of gene expression data. It also provides information about the presence of chip artifacts. Hence it is recommended to consider correlation between intensities of various types of pixels between the spots. But all the reviewed articles are describing the importance of the correlation but none of them explained how to estimate the correlation between pixels from foreground and background area of the spots.

IV Conclusion

In cdna microarray image the pixel intensities of the spots are correlated because of various chip artefacts. it is important to estimate these correlations of pixel intensities of spots to propagate the uncertainty associated with the intensity ratio as well as to address the systematic errors arise from the chip artifacts. the current review tried to identify the methodologies used to estimate these correlations between various types of pixel intensities of the spots of microarray image file and we identified that,

- There are very few studies that explain the importance of correlation between various pixel intensities of spots in microarray experiments.
- None of the studies explored the method to estimate the correlation between pixels from foreground and background area of the microarray spots.

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COST-BENEFIT ANALYSIS OF A TWO-UNIT CENTRIFUGE SYSTEM CONSIDERING REPAIR AND REPLACEMENT

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ABSTRACT :

The present paper is an attempt to analyse a two-unit cold standby centrifuge system having identical units considering minor/major faults with on-line repairs and replacement. It is assumed that on occurrence of a minor fault the system leads to partial failure state and an on-line repair is start for the system by the repairman whereas on occurrence of a major fault it leads to complete failure state. On complete failure of the system, the repairman first inspect the system and check whether the fault is repairable or non repairable and then accordingly carry out the repair or replacement of the component of the system involved. Various measures of system effectiveness such as MTSF, Expected Uptime with full / reduced capacity, Busy Period and Profit are obtained by using Markov processes and regenerative point technique. The analysis of the system is carried out on the basis of the graphical studies and conclusions are drawn regarding the reliability, availability and profit of the system.

Keywords: ***Centrifuge System, MTSF, Expected Uptime, Busy Period, Profit, Markov Process, Regenerative Point Technique***

INTRODUCTION

In the present scenario filtration and purification plays a very important role in the modern society pertaining to the health of the human being and the qualities of the products used by them. A large number of equipments or systems of equipments are involved in the industries to meet out the requirements of such products. One such system is a centrifuge system used for separation of two objects having different type of density. Centrifuge system is being used in Refineries for oil purification, in milk plants to extract the fats, in laboratories for blood fractionation and wine clarification etc. The centrifuge system works using the sedimentation principle, where the centripetal acceleration causes more dense substances to separate out along the radial direction and lighter objects will tend to move outward direction. Thus the reliability and cost of the centrifuge system plays a very significant role in such type of industries and hence need to be analyzed.

The working of a centrifuge system in Jindal Drilling and Industries Ltd., BKC Bandra East Mumbai was observed and the real data on failure/faults, inspection, maintenance, repairs and replacement etc. for the system was collected. It was found that the system can have various kinds of faults that lead to failure/degradation of the system. Further some major faults are repairable and others non repairable. The sample of the data collected regarding types of faults / failures, maintenances, repairs and replacement on the two-unit centrifuge system working at Jindal Drilling and Industries Ltd., BKC Bandra East Mumbai, India is as follows-

Data on Failure/ Repair/ Replacement of the Centrifuge System				
Date of Failure	**Type of Fault**	**Category of Faults**	**Maintenance /Repair/ Replacement**	**Time(per hour) Maintenance/ Repair/ Replacement**
08/02/2009	Alignment	Minor	Maintenance	3.0
12/02/2009	Bearing Damage	Major	Replacement	4.2
13/03/2009	Liquid Seal Broken	Minor/Neglected	Repair	2.0
25/03/2009	Abnormal Sound	Minor/Neglected	Repair	3.0
08/04/2009	Vibrations	Minor/Neglected	Maintenance	1.0
21/04/2009	O-Ring Damage	Major	Replacement	2.5

Table-1 Sample of Data collected from Jindal Drilling and Industries Ltd.

In fact a large number of researchers in the field of reliability modeling including Gupta and Kumar (1983), Gopalan and Murlidhar (1991), Tuteja et al (2001), Taneja et al (2004), Taneja and Parashar (2007), Gupta et al (2008), Kumar et al (2010), etc. analyzed various one-unit/ two-unit systems. Tuteja et al. (2001) studied reliability and profit analysis of two-unit cold standby system with partial failure and two types of repairman. Taneja and Nanda (2003) studied probabilistic analysis of a two-unit cold standby system with resume and repeat repair policies. Kumar and Bhatia (2011, 2012, 2013) discussed the behaviour of the single unit centrifuge system considering the concepts of inspections, halt of system, degradation, minor/major faults, neglected faults, online/offline maintenances, repairs of the faults etc. Recently, Kumar V. et al. (2014) discussed the profit analysis of a two-unit cold standby centrifuge system with single repairman.

As far as we concern with the research work on reliability modeling, none of the researchers have analyzed such a two-unit cold standby centrifuge system considering such type of various faults. To fill up this gap, the present paper analyse a two unit cold standby centrifuge system considering minor and major faults with on-line repair and replacement after an inspection. Whereas faults such as alignment disturbed, liquid seal broken, motor over heating (overload), abnormal sound, wear and tear friction pad, under/over voltage, vibration etc. are considered as minor faults and are repaired on-line and faults such as bearing damage, gear damaged, o-ring damage, motor damage or motor burnt etc. are considered as major faults and are repaired or replaced off-line. It is assumed that minor fault leads to down state while major fault leads to complete failure of the system. On complete failure of the system, the repairman first inspect whether the fault is repairable or non repairable and accordingly carry out the repair or replacement of the components involved. Various measures of system effectiveness such as mean sojourn time, MTSF, expected up time with full / reduced capacity of the system and busy period of the repairman are obtained using Markov processes and regenerative point technique. The conclusions regarding reliability, availbility and profit of the system are given on the basis of graphical studies.

SYSTEM DESCRIPTION AND OTHER ASSUMPTIONS

1. The system consist two identical units.
2. Each unit of the system has three modes i.e. Operative, Partially failed and Failed.
3. Initially the system starts operation from zero (0^{th}) state in which both the units are operative mode.
4. Faults are self- announcing on occurring in the system.
5. There is a single repairman facility with the system to repair the fault.
6. After each repair the system is as good as new.

7. Inspection is carried out only on the occurrence of major faults.
8. During online repair/waiting for repair there may be occurrence of major fault.
9. The failure time distributions are exponential while other time distributions are general.
10. Switching is perfectly done on occurrence of major fault.
11. All the random variables are mutually independent.

NOTATIONS

λ_1/ λ_2	Rate of occurrence of major/ minor failure
a/ b	Probability that a fault is repairable/ non-repairable
$i_1(t)$/ $I_1(t)$	p.d.f./ c.d.f. of time to inspection of the unit at failed state
$g_1(t)$/ $G_1(t)$	p.d.f./ c.d.f. of times to repair of minor fault at down state
$g_2(t)$/ $G_2(t)$	p.d.f./ c.d.f. of times to repair the unit at failed state
$h_1(t)$/ $H_1(t)$	p.d.f./ c.d.f. of times to replacement of the unit at failed state
O_r/ O_w / O_{cs}	Operative unit under repair/ waiting/ cold standby
F_i / F_r / F_{rp} / F_w	Failed unit under inspection/ repair/ replacement/ waiting

TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

A state-transition diagram in fig. 1 shows various states of transition of the system. The epochs of entry into states 0, 1, 2, 3 and 4 are regeneration points and thus these are regenerative states. The states 5, 6, 7 and 8 are failed state.

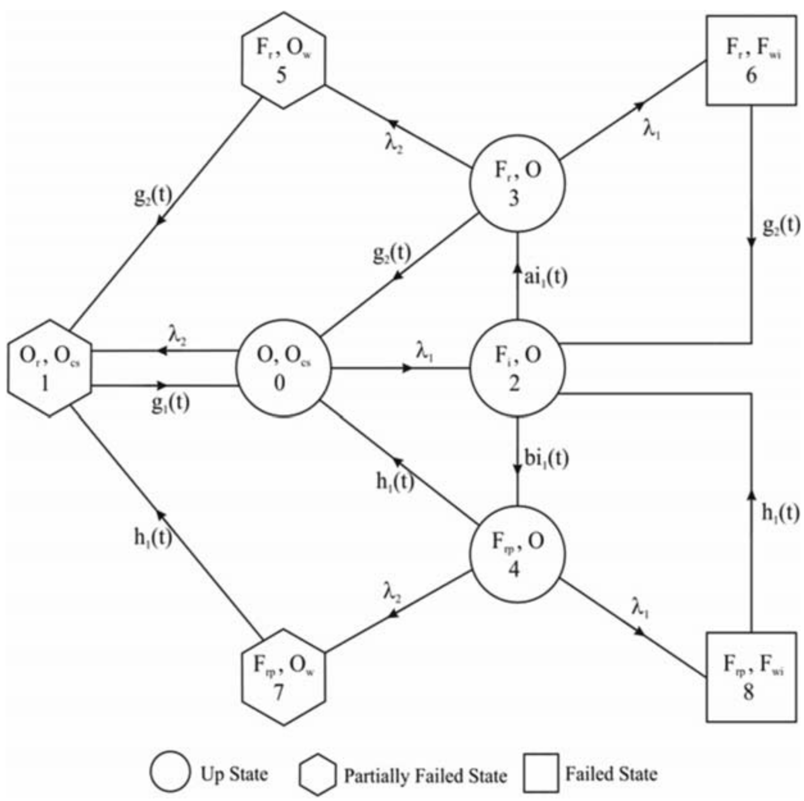

Fig.-1 State Transition Diagram

The transition probabilities are given by

$$dQ_{01}(t) = \lambda_2 e^{-(\lambda_1+\lambda_2)t}dt \qquad dQ_{02}(t) = \lambda_1 e^{-(\lambda_1+\lambda_2)t}dt$$

$dQ_{10}(t)=g_1(t)dt$ $\quad$ $dQ_{23}(t)=ai_1(t)dt$

$dQ_{24}(t)=bi_1(t)dt$ $\quad$ $dQ_{30}(t)=e^{-(\lambda_1+\lambda_2)t}g_2(t)dt$

$dQ_{31}^{5}(t)=\left(\lambda_2 e^{-(\lambda_1+\lambda_2)t}©1\right)g_2(t)dt$ $\quad$ $dQ_{32}^{6}(t)=\left(\lambda_1 e^{-(\lambda_1+\lambda_2)t}©1\right)g_2(t)dt$

$dQ_{35}(t)=\lambda_2 e^{-(\lambda_1+\lambda_2)t}\overline{G_2}(t)dt$ $\quad$ $dQ_{36}(t)=\lambda_1 e^{-(\lambda_1+\lambda_2)t}\overline{G_2}(t)dt$

$dQ_{40}(t)=e^{-(\lambda_1+\lambda_2)t}h_1(t)dt$ $\quad$ $dQ_{41}^{7}(t)=\left(\lambda_2 e^{-(\lambda_1+\lambda_2)t}©1\right)h_1(t)dt$

$dQ_{42}^{8}(t)=\left(\lambda_1 e^{-(\lambda_1+\lambda_2)t}©1\right)h_1(t)dt$ $\quad$ $dQ_{47}(t)=\lambda_2 e^{-(\lambda_1+\lambda_2)t}\overline{H_1}(t)dt$

$dQ_{48}(t)=\lambda_1 e^{-(\lambda_1+\lambda_2)t}\overline{H_1}(t)dt$

Taking L.S.T $Q_{ij}^{**}(s)$ and $p_{ij}=\lim_{s\to 0}Q_{ij}^{**}(s)$, the non-zero elements $p_{ij,}$ are obtained as under:

$$p_{01}=\frac{\lambda_2}{\lambda_1+\lambda_2} \qquad p_{02}=\frac{\lambda_1}{\lambda_1+\lambda_2} \qquad p_{10}=g_1^{*}(0)$$

$$p_{23}=ai_1^{*}(0) \qquad p_{24}=bi_1^{*}(0) \qquad p_{30}=g_2^{*}(\lambda_1+\lambda_2)$$

$$p_{31}^{5}=\frac{\lambda_2\left[1-g_2^{*}(\lambda_1+\lambda_2)\right]}{\lambda_1+\lambda_2}=p_{35} \qquad p_{32}^{6}=\frac{\lambda_1\left[1-g_2^{*}(\lambda_1+\lambda_2)\right]}{\lambda_1+\lambda_2}=p_{36}$$

$$p_{40}=h_1^{*}(\lambda_1+\lambda_2) \qquad p_{41}^{7}=\frac{\lambda_2\left[1-h_1^{*}(\lambda_1+\lambda_2)\right]}{\lambda_1+\lambda_2}=p_{47}$$

$$p_{42}^{8}=\frac{\lambda_1\left[1-h_1^{*}(\lambda_1+\lambda_2)\right]}{\lambda_1+\lambda_2}=p_{48}$$

By these transition probabilities, it can be verified that

$$p_{01}+p_{02}=1, \qquad p_{23}+p_{24}=1, \qquad p_{30}+p_{35}+p_{36}=1, \qquad p_{30}+p_{31}^{5}+p_{32}^{6}=1,$$

$$p_{40}+p_{47}+p_{48}=1, \qquad p_{40}+p_{41}^{7}+p_{42}^{8}=1, \qquad p_{10}=p_{51}=p_{62}=p_{71}=p_{82}=1$$

The unconditional mean time taken by the system to transit for any regenerative state j, when it is counted from epoch of entrance into that state i, is mathematically stated as-

$$m_{ij}=\int_0^{\infty} t\,dQ_{ij}(t)=-q_{ij}^{*\prime}(0), \text{ Thus-}$$

$$m_{01}+m_{02}=\mu_0 \qquad m_{10}=\mu_1 \qquad m_{23}+m_{24}=\mu_2$$

$$m_{30}+m_{35}+m_{36}=\mu_3 \qquad m_{30}+m_{31}^{5}+m_{32}^{6}=k_1$$

$$m_{40}+m_{47}+m_{48}=\mu_4 \qquad m_{40}+m_{41}^{7}+m_{42}^{8}=k_2$$

Where

$$k_1=-g_2^{*\prime}(0), \qquad k_2=-h_1^{*\prime}(0)$$

The mean sojourn time in the regenerative state i(μi) is defined as the time of stay in that state before transition to any other state then we have

$$\mu_0=\frac{1}{\lambda_1+\lambda_2} \qquad \mu_1=g_1^{*\prime}(0) \qquad \mu_2=i_1^{*\prime}(0) \qquad \mu_3=\frac{1-g_2^{*}(\lambda_1+\lambda_2)}{(\lambda_1+\lambda_2)}$$

$$\mu_4=\frac{1-h_1^{*}(\lambda_1+\lambda_2)}{(\lambda_1+\lambda_2)}$$

MEASURES OF THE SYSTEM EFFECTIVENESS

Various measures of the system effectiveness obtained in steady state using the arguments of the theory of regenerative process are as under:

The Mean Time to System Failure (MTSF) **= N/D**
Expected Up-Time of the System with Full Capacity (AF_0) **= N_1/D_1**
Expected Up-Time of the System with Reduced Capacity (AR_0) **= N_2/D_1**
Busy Period of Repair Man (Inspection Time Only) **= N_3/D_1**
Busy Period of Repair Man (Repair Time Only) **= N_4/D_1**
Busy Period of Repair Man (Replacement Time Only) **= N_5/D_1**

where

$$N = \mu_0 + p_{01}\mu_1 + p_{02}\left[\mu_2 + p_{23}\left\{\mu_3 + p_{35}\mu_1\right\} + p_{24}\left\{\mu_4 + p_{47}\mu_1\right\}\right]$$

$$D = 1 - p_{01} - p_{02}\left[p_{23}\left(p_{30} + p_{35}\right) + p_{24}\left(p_{40} + p_{47}\right)\right]$$

$$N_1 = \mu_0\left(1 - p_{23}p_{32}^{6} - p_{24}p_{42}^{8}\right) + p_{02}\mu_2 + p_{02}p_{23}\mu_3 + p_{02}p_{24}\mu_4$$

$$D_1 = \left(1 - p_{23}p_{32}^{6} - p_{24}p_{42}^{8}\right)\left(\mu_0 + p_{01}\mu_1\right) + p_{02}\left(p_{23}p_{31}^{5} + p_{24}p_{41}^{7}\right)\mu_1 + p_{02}\left(\mu_2 + p_{23}k_1 + p_{24}k_2\right)$$

$$N_2 = p_{01}\mu_0\left(1 - p_{23}p_{32}^{6} - p_{24}p_{42}^{8}\right) + p_{02}\left(p_{23}p_{31}^{5} + p_{24}p_{41}^{7}\right)\mu_1$$

$$N_3 = p_{02}\mu_2$$

$$N_4 = \left(1 - p_{23}p_{32}^{6} - p_{24}p_{42}^{8}\right)p_{01}\mu_1 + p_{02}\left(p_{23}p_{31}^{5} + p_{24}p_{41}^{7}\right)\mu_1 + p_{02}p_{23}\mu_3$$

$$N_5 = p_{02}p_{24}\mu_4$$

PROFIT ANALYSIS

The expected profit incurred of the system is-

$$P = C_0AF_0 + C_1AR_0 - C_2B_i - C_3B_r - C_4B_{rp} - C_5$$

where

C_0 = Revenue per unit uptime of the system with full capacity.
C_1 = Revenue per unit uptime of the system with reduced capacity.
C_2 = Cost per unit inspection of the failed unit
C_3 = Cost per unit repair of the failed unit
C_4 = Cost per unit replacement of the failed unit
C_5 = Cost of installation

GRAPHICAL INTERPRETATION AND CONCLUSION

For graphical analysis following particular cases are considered-

$$g_1(t) = \beta_1 e^{-\beta_1 t} \qquad g_2(t) = \beta_2 e^{-\beta_2 t} \qquad i_1(t) = \alpha_1 e^{-\alpha_1 t} \qquad h_1(t) = \gamma_1 e^{-\gamma_1 t}$$

Therefore, we have

$$p_{01} = \frac{\lambda_2}{\lambda_1 + \lambda_2} \qquad p_{02} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \qquad p_{10} = 1$$

$p_{23} = a$ $\quad p_{24} = b$ $\quad p_{30} = \dfrac{\beta_2}{\lambda_1 + \lambda_2 + \beta_2}$

$p_{31}^5 = \dfrac{\lambda_2}{\lambda_1 + \lambda_2 + \beta_2} = p_{35}$ $\quad p_{32}^6 = \dfrac{\lambda_1}{\lambda_1 + \lambda_2 + \beta_2} = p_{36}$ $\quad p_{40} = \dfrac{\gamma_1}{\lambda_1 + \lambda_2 + \gamma_1}$

$p_{41}^7 = \dfrac{\lambda_2}{\lambda_1 + \lambda_2 + \gamma_1} = p_{47}$ $\quad p_{42}^8 = \dfrac{\lambda_1}{\lambda_1 + \lambda_2 + \gamma_1} = p_{48}$

$\mu_0 = \dfrac{1}{\lambda_1 + \lambda_2}$ $\quad \mu_1 = \dfrac{1}{\beta_1}$ $\quad \mu_2 = \dfrac{1}{\alpha_1}$

$\mu_3 = \dfrac{1}{\lambda_1 + \lambda_2 + \beta_2}$ $\quad \mu_4 = \dfrac{1}{\lambda_1 + \lambda_2 + \gamma_1}$

Various graphs are plotted for MTSF, Expected up time and Expected down time and Profit of the system by taking different values of failure rates (λ_1 & λ_2), inspection rate (α_1), repair rates (β_1& β_2), replacement rate (γ_1) and probabilities of repairable & non-repairable (a & b).

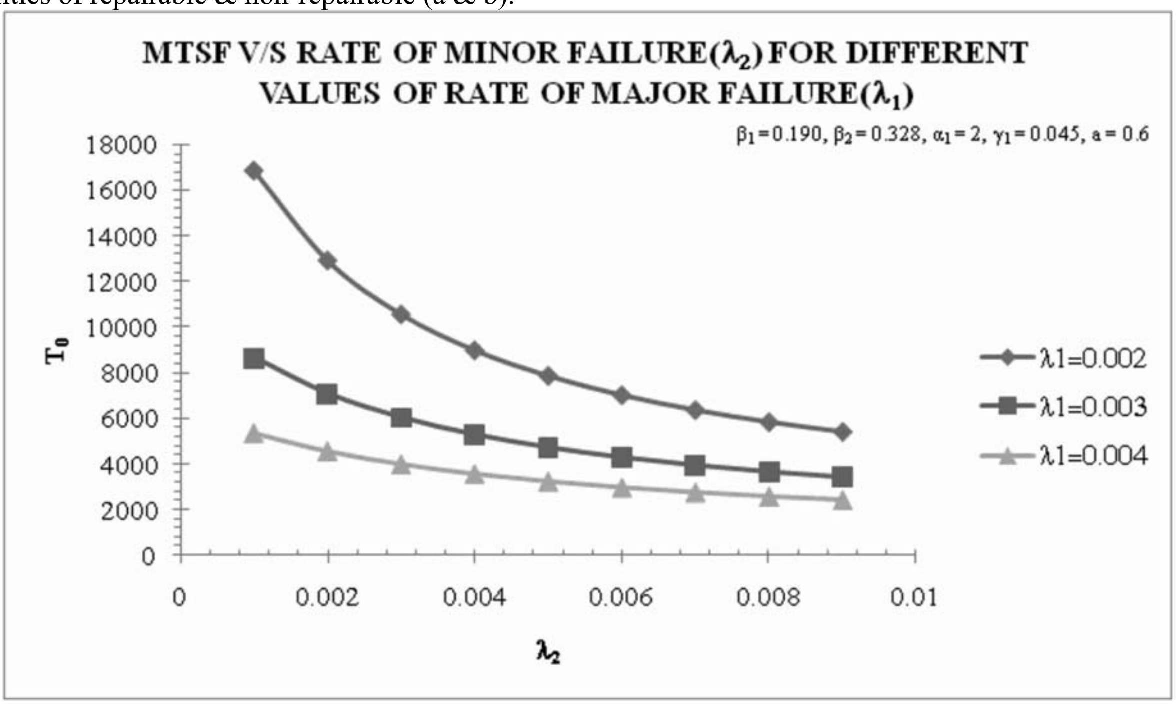

Fig.2

Fig.2 gives the graph between MTSF (T_0) and the rate of failure (λ_2) due to minor faults for different values of the rate of failure (λ_1) due to major faults. The graph reveals that the MTSF decreases with increase in the values of the failure rates.

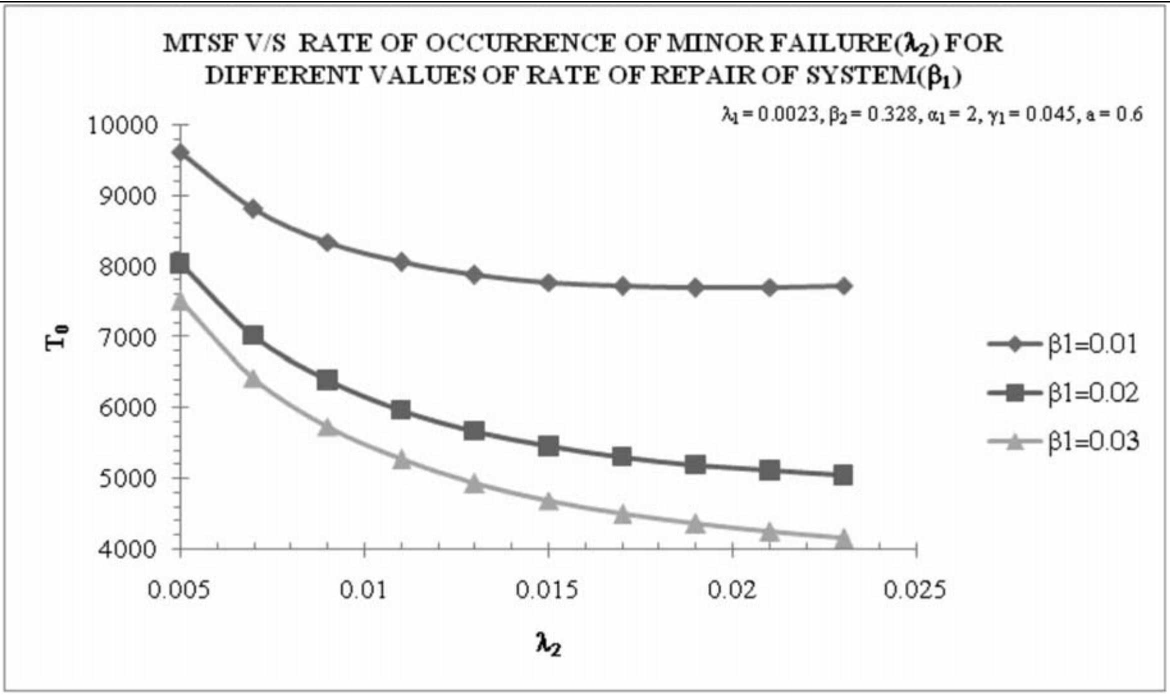

Fig.3

The curves in Fig.3 give the graph between MTSF (T_0) and rate of occurrence of minor failure (λ_2) for different values of rate of repair (β_1) of the system. The graph reveals that the MTSF decreases with increase in the values of the failure rate as well as the repair rate.

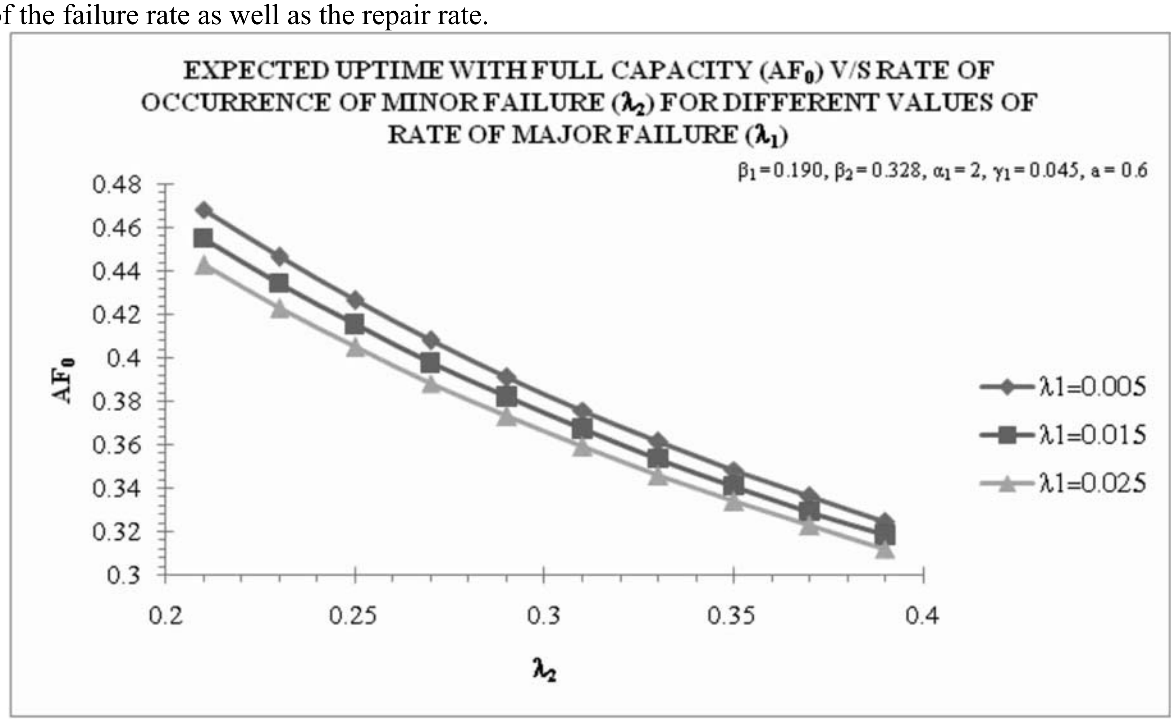

Fig.4

Fig. 4 gives the graph between Expected uptime with full capacity (AF_0) and the rate of occurrence of minor faults (λ_2) for different values of rate of occurrence of major faults (λ_1). The graph reveals that the Expected uptime with full capacity decreases with increase in the values of the failure rates.

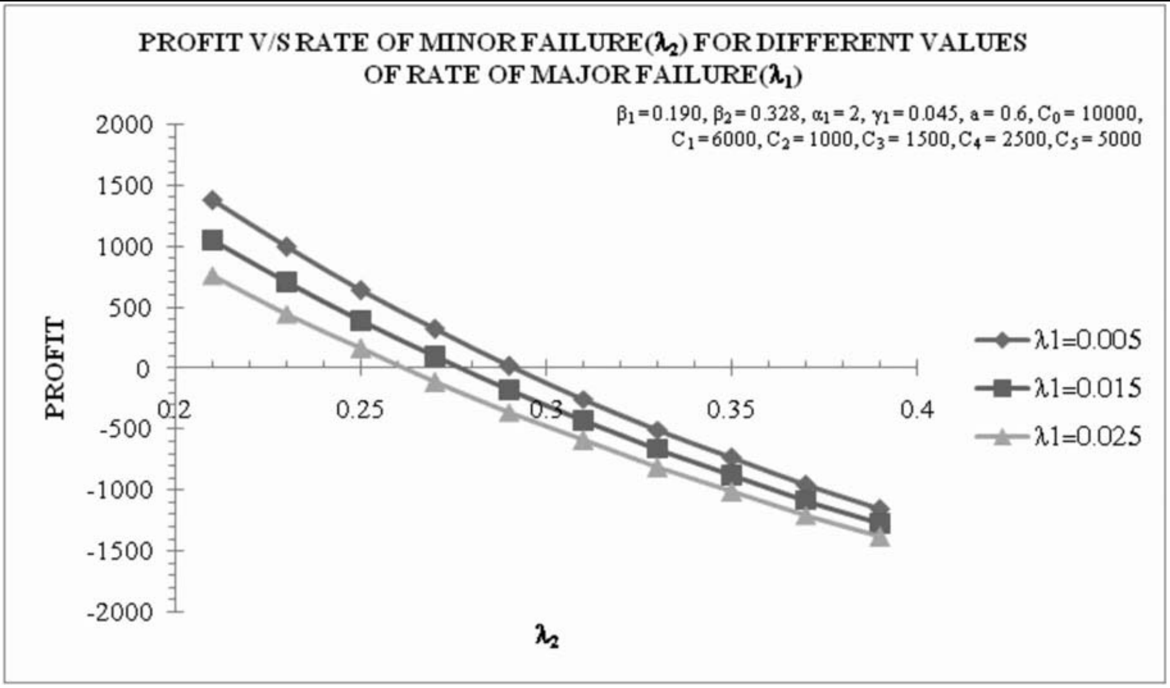

Fig.5

The curves in the Fig.5 show the behavior of the profit with respect to the rate of occurrence of minor faults (λ_2) for the different values of rate of occurrence of major faults (λ_1). It is evident from the graph that profit decreases with the increase in the values of the rate of occurrence of minor faults and has lower values for higher values of the rate of occurrence of major faults when other parameters remain fixed. From the Fig.6 it may also be observed that for $\lambda_1 = 0.005$, the profit is positive or zero or negative according as λ_2 is < or = or > 0.2913. Hence the system is profitable to the industry whenever $\lambda_2 < 0.2913$. Similarly, for $\lambda_1 = 0.015$ and $\lambda_1 = 0.025$ respectively the profit is negative or zero or positive according as λ_2 is < or = or > 0.2765 and 0.2614 respectively. Thus, in these cases, the system is profitable to the industry whenever $\lambda_2 < 0.2765$ and 0.2614 respectively.

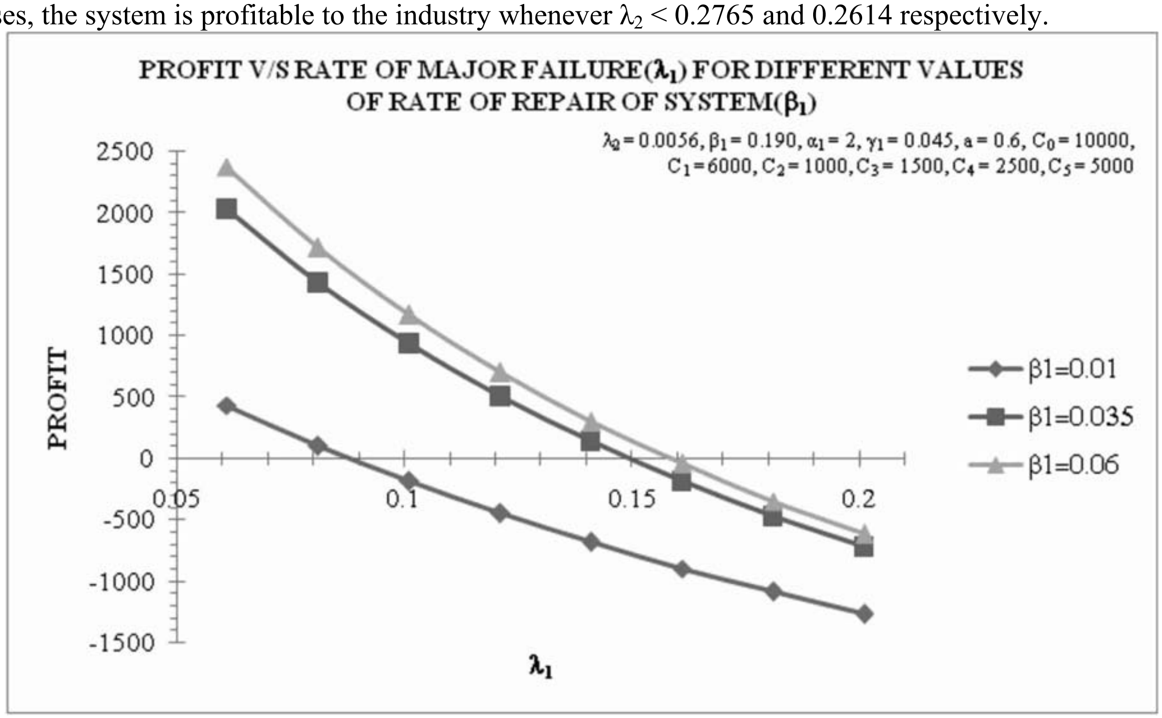

Fig.6

The curves in the Fig.6 show the behavior of the profit and the rate of failure of occurrence of major faults (λ_1) for different values of rate of repair (β_1) of the system. It is evident from the graph that profit decreases with the increase in the values of the rate of occurrence of minor faults and has higher values for higher values of the repair rate when other parameters remain fixed. From the Fig.6 it may also be observed that for $\beta_1 = 0.01$, the profit is positive or zero or negative according as λ_1 is < or = or > 0.088. Hence the system is profitable to the industry whenever $\lambda_1 < 0.088$. Similarly, for $\beta_1 = 0.035$ and 0.06 respectively the profit is positive or zero or negative according as λ_1 is < or = or > 0.149 and 0.158 respectively. Thus, in these cases, the system is profitable to the industry whenever $\lambda_1 < 0.149$ and 0.158 respectively.

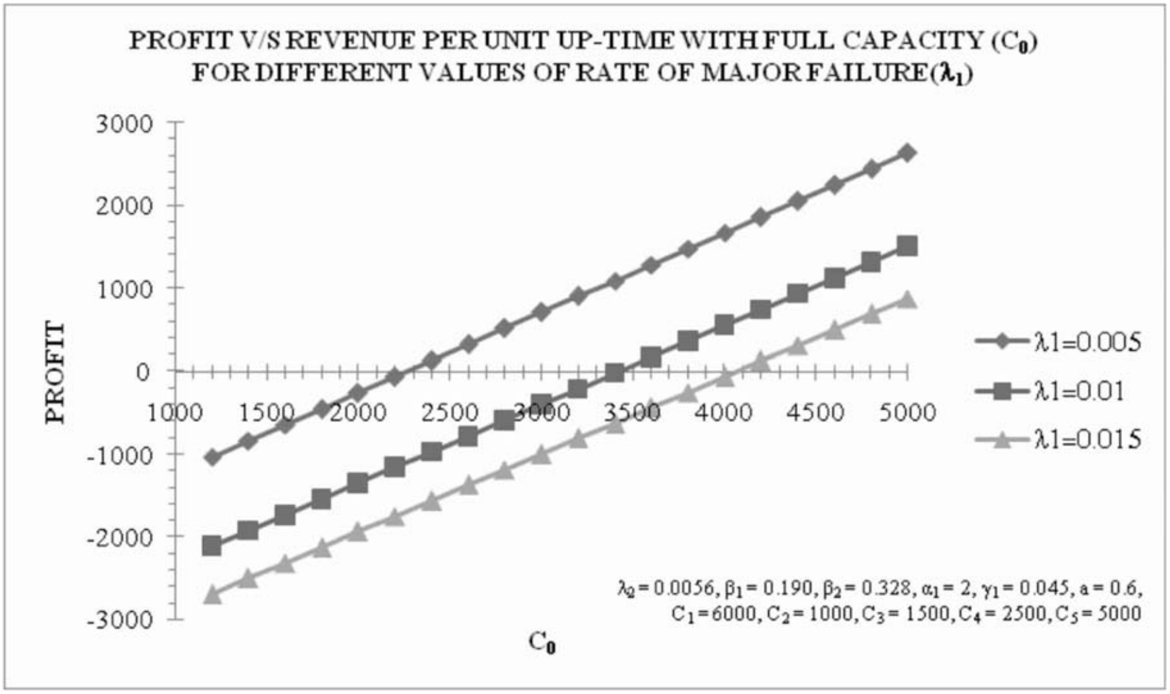

Fig.7

The curves in the Fig.7 show the behavior of the profit with respect to the revenue per unit up time with full capacity (C_0) of the system for the different values of rate of occurrence of major faults (λ_1). It is evident from the graph that profit increases with the increase in revenue per unit up time of the system with full capacity for fixed value of the rate of occurrence of major faults. From the Fig.7 it may also be observed that for $\lambda_1 = 0.005$, the profit is negative or zero or positive according as C_0 is < or = or > Rs. 2269.14. Hence the system is profitable to the industry whenever $C_0 \geq$ Rs. 2269.14. Similarly, for $\lambda_1 = 0.01$ and $\lambda_1 = 0.015$ respectively the profit is negative or zero or positive according as C_0 is < or = or > Rs. 3422.98 and Rs. 4069.01 respectively. Thus, in these cases, the system is profitable to the industry whenever $C_0 \geq$ Rs. 3422.98 and Rs. 4069.01 respectively.

CONCLUSION

The analysis performed above concludes that the mean time to system failure (MTSF) decreases with increase in the values of the failure rates. The expected uptime with full capacity decreases with increase in the values of the failure rates. The profit of the two-unit cold standby centrifuge system decreases with the increase in the values of the rate of occurrence of minor/major faults whether the results are different for repair rate i.e. as the repair rate of the system is increased the performance of the system is also increased which leads to gain the profit for the system. For the profit of the system, the analysis stated various cut-off points for the revenue per unit uptime with full capacity to enhance the profit of the system.

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AN EOQ MODEL FOR DETERIORATING ITEMS WITH RAMP TYPE DEMAND,WEIBULL DISTRIBUTED DETERIORATION AND SHORTAGES.

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ABSTRACT :
In this paper, an inventory model is developed for deteriorating items that deteriorates at a Weibull distributed rate, assuming the demand rate as a ramp type function of time. Shortages are allowed and completely back logged. The deterioration rate is a special form of two parameters Weibull distribution function considered by Covert and Philip [4]. Deterioration rate is considered to be a function of time and some other parameters such as temperature, humidity etc. Model is solved by cost minimizing criterion. Finally numerical examples are given to illustrate the results obtained.
Keywords: ***EOQ Model, Deterioration, Shortages, Weibull Distribution, Ramp Type Demand.***

1. INTRODUCTION

The effect of deterioration is very important in many inventory models. Deterioration is defined as decay, damage or spoilage that prevents the items from being used for its original purpose. Food items, drugs, pharmaceuticals, radioactive substances, blood and agricultural products are few examples of items in which sufficient deterioration can take place during the normalstorageperiod of the units and consequently this loss must be taken into account when analyzing the model. Efforts in analyzing inventory models inwhich a constant or variable proportion of the on-hand inventory deteriorates with time have been developed by Ghareand Schrader[8],Covert and Philip[4],Shah[18],Misra[14]etc.In their investigations, the market demand for the items is considered to be constant. Inventory models with time dependent demand have been developed by Donaldson [7], Silver [19], Mandal and Pal [11] Hari Kishan etal. [24] & Sanjay etal [25] etc. Inventory models with ramp type demand rate have been studied by Jain and Kumar[9],Jalan,Giri and Chaudhary[10],Mandal and Pal [12], Skouri,Konstantara, Papachristos and Ganas[20], Wu[22], Wu and Ouyang[23],Wu,Lin, Tan and Lee[21] etc.

In the present paper attempts have been made to develop and solve an inventory model for deteriorating items taking deterioration rate as a time dependent random function and ramp type demand rate. There are certain food items or vegetables or pharmaceuticals in which deterioration depends upon the fluctuations of humidity, temperature, etc. Hence it would be reasonable if we assume the deterioration function to depend upon a parameter α in addition to time 't' which ranges over a space 'Γ' in which probability density function is p(α).As a special case ,when there is no deterioration ,our result matches with those obtained by Wu,Lin, Tan and Lee[21].

2. ASSUMPTIONS AND NOTATIONS:

(i) The demand rate D(t) is assumed to be a ramp type function of time as $D(t)= D_o[\ t-(t-\mu)H(t-\mu)\ ]$ where $D_o > 0$ and $H(t-\mu)$ is well known Heaviside unit function defined by

$$H(t-\mu) = \begin{cases} 0 \text{ if } t \le \mu \\ 1 \text{ if } t > \mu \end{cases}$$

(ii) The deterioration rate $\theta(t,\alpha)$ is assumed in the form $\theta(t,\alpha)=\theta_0(\alpha)\, t$, $0 < \theta_0(\alpha) << 1$, $t > 0$

which is a special form of the two parameter Weibulldensity function considered by Covert and Philip[4].The function$\theta_0(\alpha)$ is some function of the random variable α, which ranges over a space 'Γ' with probability density function p(α) such that $\int_\Gamma p(\alpha)d\alpha = 1$.

(iii) Lead time is zero.
(iv) Replenishmentsize is constant and replenishment rate is infinite.
(v) T is the fixed length of each production cycle.
(vi) C_1 is the inventory holding cost per unit per unit time.
(vii) C_2 is the shortage cost per unit per time.
(viii) C_3 is the cost of each deteriorated item.
(ix) Shortages are allowed and fully backlogged.

3. MATHEMATICAL MODEL AND ANALYSIS FOR THE SYSTEM:

Let Q be the total amount of inventory produced or purchased at the beginning of each period and after fulfilling the backorders, let us assume that we get an amount S (> 0) as the initial inventory. Due to market demand and deterioration of items ,the level of inventory gradually diminishes during the period (o,t_1) and ultimately falls to zero at $t=t_1$. Shortages occurduring the period (t_1,T) which are fully backlogged. Let I(t) be the on- hand inventory at any time t. We consider two cases:

Case 1 : When μ ⍰ t_1.

The differential equations which the on-hand inventoryI(t) at any time t must satisfy inthree different parts of the cycle time T are

$$\frac{dI(t)}{dt} + \theta_0(\alpha)\, t\, I(t) = - D_o t \quad 0 \le t \le \mu \tag{1}$$

$$\frac{dI(t)}{dt} + \theta_0(\alpha)\, t\, I(t) = - D_o\mu \quad \mu \le t \le t_1 \tag{2}$$

$$\frac{dI(t)}{dt} = -D_o\mu \quad t_1 \le t \le T \tag{3}$$

Solution ofequation (1), using initial condition I(0) = S , is given by

$$I(t)e^{\frac{\theta_0(\alpha)t^2}{2}} = -D_o\left[\frac{t^2}{2} + \frac{\theta_0(\alpha)t^4}{8}\right] + S \tag{4}$$

i.e.,

$$I(t) = -D_o\left[\frac{t^2}{2} + \frac{\theta_0(\alpha)t^4}{8}\right]e^{\frac{-\theta_0(\alpha)t^2}{2}} + Se^{\frac{-\theta_0(\alpha)t^2}{2}} \tag{5}$$

$$= -D_o\left[\frac{t^2}{2} + \frac{\theta_0(\alpha)t^4}{8}\right]\left[1 - \frac{\theta_0(\alpha)t^2}{2}\right] + S\left[1 - \frac{\theta_0(\alpha)t^2}{2}\right]$$

So

$$I(t) = -D_o\left[\frac{t^2}{2} - \frac{\theta_0(\alpha)t^4}{8}\right] + S\left[1 - \frac{\theta_0(\alpha)t^2}{2}\right] \quad 0 \le t \le \mu \tag{6}$$

Solution of equation(2), using $I(t_1) = 0$, is

$$I(t)e^{\frac{\theta_0(\alpha)t^2}{2}} = -D_o\mu\left[(t - t_1) + \frac{\theta_0(\alpha)}{6}(t^3 - t_1^3)\right] \tag{7}$$

which gives

$$I(t) = -D_o\mu\left[(t - t_1) - \frac{\theta_0(\alpha)}{2}(t^3 - t^2 t_1) + \frac{\theta_0(\alpha)}{6}(t^3 - t_1^3)\right] \quad \mu \le t \le t_1 \tag{8}$$

Using continuity of I(t) at $t = \mu$, we have

$$-D_o\mu\left[(\mu - t_1) + \frac{\theta_0(\alpha)}{6}(\mu^3 - t_1^3)\right] = -D_o\left[\frac{\mu^2}{2} + \frac{\theta_0(\alpha)\mu^4}{8}\right] + S$$

which implies

$$S = D_o\left[\frac{\mu^2}{2} + \frac{\theta_0(\alpha)\mu^4}{8} - \mu(\mu - t_1) - \frac{\theta_0(\alpha)}{6}\mu(\mu^3 - t_1^3)\right]$$

$$= D_0\mu\left[-\frac{\mu}{2} + t_1 + \frac{\theta_0(\alpha)}{24}(4t_1^3 - \mu^3)\right] \tag{9}$$

Total amount of deteriorated items is

$$D = S - \int_0^{t_1} D(t)dt$$

$$= S - \int_0^{\mu} D(t)dt - \int_{\mu}^{t_1} D(t)dt$$

$$= S - \int_0^{\mu} D_0 t\, dt - \int_{\mu}^{t_1} D_0\mu\, dt$$

$$= S - D_0\frac{\mu^2}{2} - D_0\mu(t_1 - \mu)$$

$$= D_0\mu\,\frac{\theta_0(\alpha)}{24}(4t_1^3 - \mu^3) \tag{10}$$

The total number of inventory carried during the interval $(0,t_1)$ is

$$I_1 = \int_0^{t_1} I(t)dt$$
$$= \int_0^{\mu} I(t)\,dt + \int_{\mu}^{t_1} I(t)\,dt$$
$$= \int_0^{\mu} [-D_0\,(\frac{t^2}{2} - \frac{\theta_0(\alpha)t^4}{8}] + S\,(1 - \frac{\theta_0(\alpha)t^2}{2})]\;dt$$
$$- D_0\mu \int_{\mu}^{t_1} [(t - t_1) - \frac{\theta_0(\alpha)}{2}(t^3 - t^2 t_1) + \frac{\theta_0(\alpha)}{6}(t^3 - t_1^3)]\;dt$$
$$= - D_0\mu\,[\frac{\mu^2}{6} + \frac{\theta_0(\alpha)\mu^4}{48} - \frac{t_1^2}{2} - \frac{\theta_0(\alpha)t_1^4}{12}] \qquad (11)$$

The total shortage quantity during the interval (t_1,T) is

$$I_2 = -\int_{t_1}^{T} I(t)dt$$
$$= D_0\mu \int_{t_1}^{T}(t - t_1)dt$$
$$= D_0\mu\,\frac{(T - t_1)^2}{2} \qquad (12)$$

The average total cost per unit time is given by

$$C(t_1, \alpha) = \frac{C_1 I_1}{T} + \frac{C_2 I_2}{T} + \frac{C_3 D}{T}$$
$$= -\frac{C_1 D_0\mu}{T}[\frac{\mu^2}{6} + \frac{\theta_0(\alpha)\mu^4}{48} - \frac{t_1^2}{2} - \frac{\theta_0(\alpha)t_1^4}{12}] + \frac{C_2 D_0\mu}{T}\frac{(T - t_1)^2}{2} + \frac{C_3}{T} D_0\mu\,\frac{\boldsymbol{\theta_0(\alpha)}}{24}(4\,t_1^3 - \mu^3)$$

The mean average total cost is given by

$$C(t_1) = \int_{\Gamma} C(t_1, \alpha)p(\alpha)\,d\alpha$$
$$= -\frac{C_1 D_0\mu}{T}[\frac{\mu^2}{6} + \frac{A\mu^4}{48} - \frac{t_1^2}{2} - \frac{At_1^4}{12}] + \frac{C_2 D_0\mu}{T}\frac{(T - t_1)^2}{2} + \frac{C_3}{T} D_0\mu\,\frac{\mathbf{A}}{24}(4\,t_1^3 - \mu^3)$$

where $A = \int_{\Gamma} \theta_0(\alpha)\,p(\alpha)\,d\alpha$.

Differentiating w.r.t. t_1, we have

$$\frac{dC(t_1)}{dt_1} = \frac{C_1 D_0\mu}{T}[t_1 + \frac{At_1^3}{3}] - \frac{C_2 D_0\mu}{T}(T - t_1) + \frac{C_3}{T} D_0\mu\,\frac{\mathbf{A}}{2}t_1^2$$
$$= \frac{D_0\mu}{T}[C_1\,(t_1 + \frac{At_1^3}{3}) - C_2(T - t_1) + \frac{\mathbf{A}}{2}C_3 t_1^2]$$

For maxima or minima,

$$\frac{dC(t_1)}{dt_1} = 0$$

which implies

$$[C_1\,(t_1 + \frac{At_1^3}{3}) - C_2(T - t_1) + \frac{\mathbf{A}}{2}C_3 t_1^2] = 0$$

Let $f(t_1) = C_1\,(t_1 + \frac{At_1^3}{3}) - C_2(T - t_1) + \frac{\mathbf{A}}{2}C_3 t_1^2$.

Then $\frac{df(t_1)}{dt_1} = C_1\,(1 + 2At_1^2) + C_2 + A\,C_3 t_1 > 0$.So $f(t_1)$ increases from negative to positive since $f(0) = -C_2 T < 0$ *and* $f(T) = C_1\left(T + \frac{AT^3}{3}\right) + \frac{\mathbf{A}}{2}C_3 T^2 > 0$.

So there exists aunique point , say $t_1^{\#} \in (0, T)$ such that $f(t_1^{\#}) = 0$.

Also $\frac{d^2C(t_1)}{dt_1^2} = \frac{D_0\mu}{T}[C_1\,(1 + A\,t_1^2) + C_2 + AC_3 t_1] > 0$ at $t_1 = t_1^{\#}$.

Hence $C(t_1)$ is minimum at $t_1 = t_1^{\#}$.

Since $\mu < t_1$, we must check weather $t_1^{\#} > \mu$ *or* not.

If $f(\mu) \le 0$, then certainly $t_1^{\#}$ lies in (μ, T).

On the other hand , if $f(\mu) > 0$ then $\frac{dC(t_1)}{dt_1} > 0$ for $t_1 \in (\mu, T)$ giving that μ is the minimum point.

Thus for the inventory model under the condition μ ⍰ t_1, the optimal point is t_1^* is max $(\mu, t_1^{\#})$.

The optimum value of S is given by $S^* = D_0\mu\,[-\frac{\mu}{2} + t_1^* + \frac{\theta_0(\alpha)}{24}(4\,t_1^{*3} - \mu^3)]$

The mean optimum value of S is

$< S^* > = D_0\mu\,[-\frac{\mu}{2} + t_1^* + \frac{A}{24}(4\,t_1^{*3} - \mu^3)]$

Total optimal amount of backorder at the end of each cycle is $D_0\mu(T - t_1^*)$.

The optimum value of Q is given by

$$Q^* = D_0\mu\,(T - t_1^*) + D_0\mu\,[-\frac{\mu}{2} + t_1^* + \frac{\theta_0(\alpha)}{24}(4\,t_1^{*3} - \mu^3)]$$

The mean optimum value of Q is given by

$<Q^*> = D_0\mu\,(T - t_1^*) + D_0\mu\,[-\frac{\mu}{2} + t_1^* + \frac{A}{24}(4\,t_1^{*3} - \mu^3)]$

Special case:If there is no deterioration ,$\theta_0(\alpha) = 0$ so that A = 0.

Then $(C_1 + C_2)\,t_1 = C_2 T$ which gives $t_1 = \frac{C_2 T}{C_1 + C_2}$.

The expressions for $< S^* >$, $< Q^* >$ and $C(t_1^*)$ are given by

$< S^* > = D_0\mu\,[-\frac{\mu}{2} + \frac{C_2 T}{C_1 + C_2}]$,

$< Q^* > = D_0\mu\,(T - \frac{\mu}{2})$,

$C(t_1^*) = -\frac{C_1 D_0\mu}{T}[\frac{\mu^2}{6} - \frac{t_1^{*2}}{2}] + \frac{C_2 D_0\mu}{T}\frac{(T - t_1^*)^2}{2}$.

Numerical Examples:

Example1.

C_1 = \$ 3 per unit per year

C_2 = \$ 15 per unit per year

C_3 = \$ 5 per unit per year

$D_0 = 100$, μ = 0.12 year , T = 1 year.

$\theta_0(\alpha) = a + b\,\alpha$ where a= 0.11 , b= 0.1 ,

$$p(\alpha) = \begin{cases} \frac{1}{2}(\alpha + 1) & -1 \le \alpha \le 1 \\ 0 & elsewhere \end{cases}$$

$A = \int_\Gamma \theta_0(\alpha)\, p(\alpha)\, d\alpha$

$= \int_{-1}^{1}(a + b\,\alpha)\frac{1}{2}(\alpha + 1)d\alpha$

$= a + \frac{b}{3} = 0.11 + \frac{0.1}{3} = 0.14$

Note that in this case $f(\mu) = -12.83 < 0$.

Equation determining $t_1^{\#}$ is $0.14\,t_1^3 + 0.35 t_1^2 + 18 t_1 - 15 = 0$ which gives $t_1^{\#} = 0.82$

Since $t_1^{\#} > \mu$, optimal value of t_1 is $t_1^* = 0.82$

Example 2.

C_1 = \$ 3 per unit per year

C_2 = \$ 15 per unit per year

C_3 = \$ 5 per unit per year

$D_0 = 100$, μ = 0.9 year , T = 1 year.

$\theta_0(\alpha) = a + b\,\alpha$ where a= 0.11 , b= 0.1 ,

$$p(\alpha) = \begin{cases} \frac{1}{2}(\alpha + 1) & -1 \le \alpha \le 1 \\ 0 & elsewhere \end{cases}$$

$A = \int_\Gamma \theta_0(\alpha)\, p(\alpha)\, d\alpha$

$= \int_{-1}^{1}(a + b\,\alpha)\frac{1}{2}(\alpha + 1)d\alpha$

$= a + \frac{b}{3} = 0.11 + \frac{0.1}{3} = 0.14$

Note that in this case f (μ) = 1.585 > 0.

Equation determining $t_1^{\#}$ is $0.14\, t_1^3 + 0.35 t_1^2 + 18 t_1 - 15 = 0$ which gives $t_1^{\#} = 0.82$

Since $t_1^{\#} < \mu$, optimal value of t_1 is $t_1^* = 0.9$

Case 2 : When $\mu \geq t_1$.

The differential equations which the on-hand inventory I(t) at any time t must satisfy in three different parts of the cycle time T are

$$\frac{dI(t)}{dt} + \theta_0(\alpha)\, t\, I(t) = -D_o t \qquad 0 \le t \le t_1 \qquad (13)$$

$$\frac{dI(t)}{dt} + \theta_0(\alpha)\, t\, I(t) = -D_o t \qquad t_1 \le t \le \mu \qquad (14)$$

$$\frac{dI(t)}{dt} = -D_o \mu \qquad \mu \le t \le T \qquad (15)$$

Solution of equation (13), using $I(t_1) = 0$, is given by

$$I(t) e^{\frac{\theta_0(\alpha) t^2}{2}} = -D_o \left[\frac{1}{2}(t^2 - t_1^2) + \frac{\theta_0(\alpha)}{8}(t^4 - t_1^4) \right]$$

i.e., $$I(t) = -D_o \left[\frac{1}{2}(t^2 - t_1^2) + \frac{\theta_0(\alpha)}{8}(t^4 - t_1^4) \right] e^{\frac{-\theta_0(\alpha) t^2}{2}}$$

$$= -D_o \left[\frac{1}{2}(t^2 - t_1^2) + \frac{\theta_0(\alpha)}{8}(t^4 - t_1^4) \right] \left[1 - \frac{\theta_0(\alpha) t^2}{2} \right]$$

$$= -\frac{D_0}{2} \left[(t^2 - t_1^2) - \frac{\theta^0(\alpha)}{8}(t^4 + t_1^4 - 2 t_1^2 t^2) \right] \qquad (16)$$

Solutions of (14) and (15) are

$$I(t) = -\frac{1}{2} D_o (t^2 - t_1^2) \quad t_1 \le t \le \mu \qquad (17)$$

$$I(t) = -\frac{1}{2} D_o (2\mu t - \mu^2 - t_1^2) \quad \mu \le t \le T \qquad (18)$$

Further amount of initial inventory S is given by

$$S = D_o \left[\frac{1}{2} t_1^2 + \frac{\theta_0(\alpha)}{8} t_1^4 \right] \qquad (19)$$

Total amount of deteriorated items is

$$D = S - \int_0^{t_1} D(t)dt$$

$$= S - \int_0^{t_1} D_0 t\,dt$$

$$= S - D_0 \frac{t_1^2}{2}$$

$$= D_o \left[\frac{1}{2} t_1^2 + \frac{\theta_0(\alpha)}{8} t_1^4 \right] - D_0 \frac{t_1^2}{2}$$

$$= D_o \frac{\theta_0(\alpha)}{8} t_1^4 \qquad (20)$$

Total amount of inventory carried during the interval $(0,t_1)$ is $I_1 = \int_0^{t_1} I(t)dt$

$$I_1 = -\frac{D_0}{2} \int_0^{t_1} \left[(t^2 - t_1^2) - \frac{\theta_0(\alpha)}{8}(t^4 + t_1^4 - 2 t_1^2 t^2) \right] dt$$

$$= \frac{D_0}{3} \left[t_1^3 + \frac{\theta_0(\alpha)}{10} t_1^5 \right]$$

The total shortage quantity during the interval $(t_1$,T) is

$$I_2 = -\int_{t_1}^{T} I(t)dt$$

$$= -\int_{t_1}^{\mu} I(t)dt - \int_{\mu}^{T} I(t)dt$$

$$= \int_{t_1}^{\mu} \frac{1}{2} D_0 (t^2 - t_1^2) dt + \int_{\mu}^{T} \frac{1}{2} D_0 (2\mu t - \mu^2 - t_1^2)\, dt$$

$$= \frac{D_0}{6} \left[\mu^3 + 2 t_1^3 + 3\mu T^2 - 3\mu^2 T - 3 t_1^2 T \right]$$

The average total cost per unit time is given by

$$C(t_1, \alpha) = \frac{C_1 I_1}{T} + \frac{C_2 I_2}{T} + \frac{C_3 D}{T}$$

$$= \frac{C_1 D_0}{3T} \left[t_1^3 + \frac{\theta_0(\alpha)}{10} t_1^5 \right] + \frac{C_2 D_0}{6T} \left[\mu^3 + 2 t_1^3 + 3\mu T^2 - 3\mu^2 T - 3 t_1^2 T \right] + C_3 D_o \frac{\theta_0(\alpha)}{8T} t_1^4$$

The mean average total cost is given by

$C(t_1) = \int_\Gamma \; C(t_{1,}\, \alpha)p(\alpha)\, d\alpha$

$= \frac{C_1 D_0}{3T}[\, t_1^3 \; + \; \frac{At_1^5}{10}] + \frac{C_2 D_0}{6T}[\, \mu^3 \; + 2t_1^3 + 3\mu T^2 \; - 3\mu^2 T - 3t_1^2 \; T\,] + C_3 D_0 \frac{A}{8T} t_1^4$

where $A = \int_\Gamma \; \theta_0(\alpha)\; p(\alpha)\; d\alpha$.

Differentiating w.r.t. t_1, we have

$$\frac{dC(t_1)}{dt_1} = \frac{C_1 D_0}{T}\left[t_1^2 \; + \frac{At_1^4}{6}\right] + \frac{C_2 D_0}{T}[\, t_1^2 - t_1 T] + \frac{C_3}{2T} D_0\, A\, t_1^3$$

$$= \frac{D_0 t_1}{T}\left[C_1\,(t_1 \; + \frac{At_1^3}{6}) + C_2\,(t_1 - T) + \frac{C_3}{2} \;\; A\, t_1^2\right]$$

Let $g(t_1) = \left[C_1\,(t_1 \; + \frac{At_1^3}{6}) + C_2\,(t_1 - T) + \frac{C_3}{2} \;\; A\, t_1^2\right]$

Then $g(0) = -C_2\, T < 0$ and $g(T) = \left[C_1\,(T \; + \frac{AT^3}{6}) + \frac{C_3}{2} \;\; A\, T^2\right] > 0.$

Since $\frac{dg(t_1)}{dt_1} = \left[C_1\,(1 \; + \frac{At_1^2}{2}) + C_2 \; + C_3 \;\; A\, t_1\right] > 0$, $g(t_1)$ increases from $g(0) < 0$ to $g(T) > 0$

So there exists a unique point , say $t_1^\# \in (0, T)$ such that $g(t_1^\#) = 0$.

Also $\frac{d^2C(t_1)}{dt_1^2} = \frac{D_0}{T}\left[C_1\,\left(2t_1 + \frac{2}{3}A\, t_1^3\right) + \; C_2(2t_1 - T) + \frac{3}{2}AC_3 t_1^2\right]$

$$= \frac{D_0}{T}\left[C_1\,\left(t_1 + \frac{1}{6}A\, t_1^3\right) + \; C_2(t_1 - T) + \frac{1}{2}AC_3 t_1^2\right]$$

$$+ \frac{D_0}{T}\left[C_1\,\left(t_1 + \frac{1}{2}A\, t_1^3\right) + \; C_2 t_1 + AC_3 t_1^2\right] > 0 \;\; at \;\; t_1 = \; t_1^\#.$$

Hence $C(t_1)$ is minimum at $t_1 = \; t_1^\#$.

Since $\mu \geq t_1$, we must check weather $t_1^\# \leq \; \mu$ *or* not.

If $g(\mu) < 0$, then $t_1^\#$ lies in (μ, T) and so minimum point is μ.

On the other hand , if $g(\mu) > 0$ then $t_1^\#$ lies in $(0, \mu)$ giving that $t_1^\#$ is the minimum point for $C(t_1)$.

Thus for the inventory model under the condition $\mu \geq t_1$, the optimal point is t_1^* is $\min(\mu, t_1^\#)$.

The optimum value of S is given by

$$S^* = \; D_0\, [\, \frac{1}{2} t_1^{*2} + \frac{\boldsymbol{\theta_0(\alpha)}}{8} t_1^{*4}\,]$$

The mean optimum value of S is

$< \; S^* > = D_0\, [\, \frac{1}{2} t_1^{*2} + \frac{\mathbf{A}}{8} t_1^{*4}\,]$

Total optimal amount of backorder at the end of each cycle is $\frac{1}{2} D_0(\, 2\mu T - \mu^2 - t_1^{*2}\,)$.

The optimum value of Q is given by

$$Q^* = \frac{1}{2} D_0(\, 2\mu T - \mu^2 - t_1^{*2}\,) + D_0\, [\, \frac{1}{2} t_1^{*2} + \frac{\boldsymbol{\theta_0(\alpha)}}{8} t_1^{*4}\,]$$

$$= \frac{1}{2} D_0(\, 2\mu T - \mu^2 + \frac{\boldsymbol{\theta_0(\alpha)}}{4} t_1^{*4}\,]$$

The mean optimum value of Q is given by

$< Q^* > = \; \frac{1}{2} D_0(\, 2\mu T - \mu^2 + \frac{\mathbf{A}}{4} t_1^{*4}\,]$

Special case: If there is no deterioration , $\theta_0(\alpha) = 0$ so that $A = 0$.

Then $(\, C_1 \; + C_2\,)\, t_1 = C_2 T$ which gives $t_1 = \frac{C_2\, T}{C_1 + C_2}$.

The expressions for $< \; S^* >$, $< \; Q^* >$ and $C(t_1^*)$ are given by

$< \; S^* > = \frac{1}{2} D_0\, [\, \frac{C_2\, T}{C_1 + C_2}]^2$,

$< \; Q^* > = \; D_0 \mu\; (\, T \; - \frac{\mu}{2}\,)$,

$C(t_1^*) \; = \; \frac{C_1 D_0 t_1^{*3}}{3T} + \frac{C_2 D_0}{6T}\; [\, \mu^3 \; + 2t_1^{*3} + 3\mu T^2 \; - 3\mu^2 T - 3t_1^{*2} \; T\,]$

4.1 Numerical Examples:

Example 3.

C_1 = \$ 3 per unit per year

C_2 = \$ 15 per unit per year

C_3 = \$ 5 per unit per year

$D_0 = 100$, μ= 0.12 year , T = 1 year.

$\theta_0(\alpha) = a + b\,\alpha$ where a= 0.11 , b= 0.1 ,

$$p(\alpha) = \begin{cases} \frac{1}{2}(\alpha + 1) & -1 \le \alpha \le 1 \\ 0 & elsewhere \end{cases}$$

$A = \int_{\Gamma} \theta_0(\alpha)\, p(\alpha)\, d\alpha$

$= \int_{-1}^{1} (a + b\,\alpha)\frac{1}{2}(\alpha + 1) d\alpha$

$= a + \frac{b}{3} = 0.11 + \frac{0.1}{3} = 0.14$

Note that in this case g(μ) = −12.83< 0.

Equation determining $t_1^{\#}$ is $0.07t_1^3 + 0.35t_1^2 + 18t_1 - 15 = 0$ which gives $t_1^{\#} = 0.82$

Since $t_1^{\#} > \mu$, optimal value of t_1 is $t_1^* = 0.12$

Example 4.

C_1 = \$ 3 per unit per year

C_2 = \$ 15 per unit per year

C_3 = \$ 5 per unit per year

$D_0 = 100$, μ = 0.9 year , T = 1 year.

$\theta_0(\alpha) = a + b\,\alpha$ where a= 0.11 , b= 0.1 ,

$$p(\alpha) = \begin{cases} \frac{1}{2}(\alpha + 1) & -1 \le \alpha \le 1 \\ 0 & elsewhere \end{cases}$$

$A = \int_{\Gamma} \theta_0(\alpha)\, p(\alpha)\, d\alpha$

$= \int_{-1}^{1} (a + b\,\alpha)\frac{1}{2}(\alpha + 1) d\alpha$

$= a + \frac{b}{3} = 0.11 + \frac{0.1}{3} = 0.14$

Note that in this case f (μ) > 0.

Equation determining $t_1^{\#}$ is $0.07t_1^3 + 0.35t_1^2 + 18t_1 - 15 = 0$ which gives $t_1^{\#} = 0.82$

Since $t_1^{\#} < \mu$, optimal value of t_1 is $t_1^* = 0.9$

5. CONCLUSION : In the present paper , we presented that the inventory is depleted not only by demand but also by Weibull distributed deterioration .Moreover, we find that the optimal time t_1^* of this model is unique and is independent of μ and D_0.The model is solved by cost criterion and the expressions for minimum mean average total cost per unit time, the optimal initial inventory and the optimal purchase quantity are obtained.

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METHOD OF LEAST SQUARES IN REVERSE ORDER: FITTING OF LINEAR CURVE TO AVERAGE MINIMUM TEMPERATURE DATA AT GUWAHATI AND TEZPUR

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ABSTRACT :
In 2015, Rahman and Chakrabarty have developed one method by stepwise application of the principle of least squares of estimating parameters associated to a linear curve, based on the principle of elimination of parameter(s) first and then minimization of sum of squares of errors in place of the principle of least squares namely minimization of sum of squares of errors first and then elimination of parameter(s), where all values of the ratio of the difference of each pair of observed values of the dependent variable to the difference of each pair of observed values of the independent variable have been used in estimating parameters involved in the curve. In this paper, discussion has been made on how parameters can be estimated using the independent values from among the values of the said ratio. One numerical application of the method has also been discussed in fitting of linear regression of monthly mean minimum temperature on the monthly average length of day at Guwahati and also at Tezpur.

Keywords : ***Linear curve, Least squares principle, Mean minimum temperature, Monthly average length, Stepwise application***

1. INTRODUCTION:

The method of least squares, which is indispensible and is widely used method of curve fitting to numerical data, was first discovered by the French mathematician Legendre in 1805 [Mansfield (1877), Paris (1805)]. The first proof of this method was given by the renowned statistician Adrian (1808) followed by its second proof given by the German Astronomer Gauss [Hamburg (1809)]. Apart from this two proofs as many as eleven proofs were developed at different times by a number of mathematicians viz. Laplace (1810), Ivory (1825), Hagen (1837), Bassel (1838), Donkim (1844), John Herscel (1850), Crofton(1870) etc.. Though none of the thirteen proofs is perfectly satisfactory but yet it has given new dimension in setting the subject in a new light. In the method of least squares, the parameters of a curve are estimated by solving the normal equations of the curve obtained by the principle of least squares. However, for a curve of higher degree polynomial, the estimation of parameters by solving the normal equations carries a complicated calculation as the number of normal equations becomes large which leads to think of searching for some simpler method of estimation of parameter. Rahman and Chakrabarty (2015) have developed one method by stepwise application of the principle of least squares of estimating parameters associated to a linear curve, based on the principle of elimination of parameter(s) first and then minimization of sum of squares of errors in place of the principle of least squares namely minimization of sum of squares of errors first and then elimination of parameter(s). In this method, all values of the ratio of the difference of each pair of observed values of the dependent variable to the difference of each pair of observed values of the independent variable have been used in estimating parameters involved. However, all of these values are not independent. In this paper, discussion has been made on how parameters can be estimated using the independent values from among the values of the said .ratio. One numerical application of the method has been discussed in fitting of linear regression of monthly mean minimum temperature on the monthly average length of day at Guwahati and also at Tezpur.

2: ESTIMATION OF PARAMETERS IN LINEAR CURVE:

Let the theoretical relationship between the dependent variable Y and the independent variable X be

$$Y = a + b\,X \qquad (2.1)$$

Where 'a' and 'b' are the two parameters.

Let $Y_1, Y_2, \ldots, Y_n$ be n observations on Y corresponding to the observations $X_1, X_2, \ldots, X_n$ of X.

The objective here is to fit the curve given by (2.1) to the observed data on X and Y. Since the n pairs of observations

$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ may not lie on the curve (2.1), they satisfy the model

$$y_i = a + b\,x_i + \xi_i, \qquad (i = 1, 2 \ldots n) \qquad (2.2)$$

2.1 ESTIMATION OF PARAMETER: BY STEPWISE APPLICATION OF PRINCIPLES OF LEAST SQUARE AND BY SOLVING NORMAL EQUATION.

From (2.2), the following $\frac{n(n+1)}{2}$ independent equations can be obtained.

$$\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = b + \left(\frac{\xi_1 - \xi_2}{x_1 - x_2}\right)$$

$$\left(\frac{y_1 - y_3}{x_1 - x_3}\right) = b + \left(\frac{\xi_1 - \xi_3}{x_1 - x_3}\right)$$

$$\left(\frac{y_1 - y_4}{x_1 - x_4}\right) = b + \left(\frac{\xi_1 - \xi_4}{x_1 - x_4}\right)$$

$$\left(\frac{y_1 - y_n}{x_1 - x_n}\right) = b + \left(\frac{\xi_1 - \xi_n}{x_1 - x_n}\right)$$

$$\left(\frac{y_2 - y_3}{x_2 - x_3}\right) = b + \left(\frac{\xi_2 - \xi_3}{x_2 - x_3}\right)$$

$$\left(\frac{y_2 - y_n}{x_2 - x_n}\right) = b + \left(\frac{\xi_2 - \xi_n}{x_2 - x_n}\right)$$

……………………………………

$$\left(\frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right) = b + \left(\frac{\xi_n - \xi_{n-1}}{x_n - x_{n-1}}\right)$$

Now, writing

$$Z_{ij} = \left(\frac{y_i - y_j}{x_i - xj}\right)$$

and $$e_{ij} = \left(\frac{\xi_i - \xi_j}{x_i - x_j} \right)$$

we have

$$Z_{ij} = b + e_{ij} \qquad i = 1,2,---n, \quad j = 1,2,---n, \quad i \angle j$$

Now minimizing $$S = \sum_{\substack{i-1 \\ i \angle j}}^{n} \sum_{j=1}^{n} \xi_{ij}^2 = \sum_{\substack{i=1 \\ i \angle j}}^{n} \sum_{j=1}^{n} \left(Z_{ij} - b\right)^2$$

With respect to *'b'*, the LSE of *'b'* can be obtained as

$$\hat{b} = \frac{2}{n(n+1)} \sum_{\substack{i=1 \\ i \angle j}}^{n} \sum_{j=1}^{n} Z_{ij} \tag{2.3}$$

Using the value of (2.3) in (2.1) we get the value of '*a*'

$$\hat{a} = \bar{y} - \hat{b}\bar{x} \tag{2.4}$$

Here we have considered average of mean minimum temperature of two cities in the context of Assam as observed data to fit the following linear equations

Let the linear equation be

$$Y_i = a + b\,X_i \qquad (\text{ i} = 1, 2. \ldots\ldots\text{n })$$

Where Y_i = Average of mean maximum Temperature.

X_i = Length of the day.

3. NUMERICAL EXAMPLE: ON MINIMUM TEMPERATURE

Ex: 3.1: Average of mean minimum temperature of Guwahati:

X_i	10.553	11.105	11.834	12.610	13.266	13.605	13.469	12.921	12.191	11.424	10.755	10.398
Y_i	10.815	14.849	16.087	20.488	22.793	25.045	25.660	25.635	24.650	22.058	16.982	12.226

Solution: The matrix Z_{ij} where $Z_{ij} = \frac{(y_i - y_j)}{(x_i - x_j)}$ has been obtained as

7.310										
4.116	1.698									
4.703	3.747	5.671								
4.415	3.676	4.683	3.514							
4.662	4.078	5.058	4.580	6.643						
5.091	4.573	5.855	6.021	14.123	-4.522					
6.258	5.939	8.784	16.550	-8.238	-0.862	0.047				
8.446	9.025	23.986	-9.933	-1.727	0.279	0.790	1.349			
12.908	22.599	-14.563	-1.324	0.399	1.370	1.761	2.389	3.379		
30.530	-6.094	-0.829	1.890	2.314	2.829	3.197	3.995	5.340	7.587	
-9.103	3.710	2.689	3.735	3.684	3.997	4.374	5.315	6.929	9.583	13.322

The equation (2.3) which gives the estimate of the parameter '*a*' as shown below

$$\widehat{b} = \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n} Z_{ij}$$

$= 4.368181818$

The equation (2.4) gives the estimates of '*b*'

$$\widehat{a} = \bar{y} - \widehat{b}\bar{x}$$

$= -\ 32.69186782$

Now, the normal equations to estimate *'a' and 'b'* are as follows

237.288 = 12 a + 144.131 b

2909.975716 = 144.131 a + 1746.108159 b

Hence, in this case

$\widehat{a}_{(NE)}$ = -28.33260446

$\widehat{b}_{(NE)}$ = 4.005240049

Result:

$\widehat{a}_{(NE)}$ **= − 28.33260446** $\widehat{a}_{(stw)}$ **= − 32.69186782**

$\widehat{b}_{(NE)}$ **= 4.005240049** $\widehat{b}_{(stw)}$ **= 4.368181818**

Estimated value of temperature ($\widehat{y}$) by both the methods: Guwahati

Table: 3.1(a)

Length of Day (x)	Observed Temperature (y)	Estimated Temperature $\widehat{y}_{(STW)}$	Estimated Temperature $\widehat{y}_{(NE)}$	Estimates of Error $\lvert\widehat{e}_{(STW)}\rvert = \lvert(y - \widehat{y}_{(STW)})\rvert$	Estimates of Error $\lvert\widehat{e}_{(NE)}\rvert = \lvert(y - \widehat{y}_{(NE)})\rvert$
10.553	10.815	13.40555491	13.93469378	2.59055491	3.11969378
11.105	14.849	15.81679127	16.14558628	0.96779127	1.29658628
11.834	16.087	19.00119581	19.06540628	2.91419581	2.97840628
12.610	20.488	22.3909049	22.17347256	1.90290490	1.68547256
13.266	22.793	25.25643218	24.80091003	2.46343218	2.00791003
13.605	25.045	26.73724581	26.15868641	1.69224581	1.11368641
13.469	25.660	26.14317309	25.61397376	0.48317309	0.04602624
12.921	25.635	23.74940945	23.41910221	1.88559055	2.21589779
12.191	24.650	20.56063672	20.49527698	4.08936328	4.15472302
11.424	22.058	17.21024127	17.42325786	4.84775873	4.63474214
10.755	16.982	14.28792763	14.74375227	2.69407237	2.23824773
10.398	12.226	12.72848672	13.31388157	0.50248672	1.08788157
Total = 144.131	**Total = 237.288**	**Total = 237.2879998 ≅ 237.288**	**Total = 237.288**	**Sum of Absolute Deviation** $\sum\lvert\widehat{e}_{(STW)}\rvert$ **= 24.33949725**	**Sum of Absolute Deviation** $\sum\lvert\widehat{e}_{(NE)}\rvert$ **= 26.57927383**

Absolute Mean Deviation ($\bar{e}_{(STW)}$ **) = 2.2028291438**

Absolute Mean Deviation ($\bar{e}_{(NE)}$ **) = 2.214939486**

1. Test of significance for estimated temperature obtained by Stepwise Application of Principles of Least Squares (stw).

The null hypothesis to be tested is

H_0: There is no significant difference between the values of observed temperature and estimated temperature.

Under the null hypothesis H_0, the test statistic is

$$t = \frac{(\bar{y} - \bar{\widehat{y}}_{(STW)})}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \text{with } (n_1 + n_2 - 2) \text{ d.f.}$$

Where $$S^2 = \frac{1}{(n_1+n_2-2)}\left[\sum_{i=1}^{12}(y_i-\bar{y})^2+(\hat{y}_i-\bar{\hat{y}}_{(STW)})^2\right] \quad \text{and} \quad n_1=n_2=11$$

Since $\bar{y} = \bar{\hat{y}}_{(STW)} = 19.774$

$\therefore$ $|t|_{cal} = 0$ and $t_{(tab,5\%,20df)} = 1.725$

$|t|_{cal} < t_{(tab,5\%,20df)}$

Thus, the null hypothesis H_0 is accepted.

Accordingly, it can be concluded that the difference between observed temperature and the corresponding estimated temperature is insignificant.

2. Test of significance for estimated temperature obtained by solving normal equations.

The null hypothesis to be tested is

H_0: There is no significant difference between the values of observed temperature and estimated temperature.

Under the null hypothesis H_0, the test statistic is

$$t = \frac{(\bar{y}-\bar{\hat{y}}_{(NE)})}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \qquad \text{with } (n_1 + n_2 - 2) \text{ d.f.}$$

Where $$S^2 = \frac{1}{(n_1+n_2-2)}\left[\sum_{i=1}^{12}(y_i-\bar{y})^2+(\hat{y}_i-\bar{\hat{y}}_{(NE)})^2\right] \quad \text{and} \quad n_1=n_2=11$$

Here, $\bar{y} = \bar{\hat{y}}_{(NE)} = 19.774$

$\therefore$ $|t|_{cal} = 0$ and $t_{(tab,5\%,20df)} = 1.725$

$|t|_{cal} < t_{(tab,5\%,20df)}$

Thus, the null hypothesis H_0 is accepted.

Accordingly, it can be concluded that the difference between observed temperature and the corresponding estimated temperature is insignificant.

Ex: 3.2: Average of mean minimum temperature of Tezpur:

X_i	10.553	11.105	11.834	12.610	13.266	13.605	13.469	12.921	12.191	11.424	10.755	10.398
Y_i	12.112	13.984	17.600	21.139	23.164	24.761	25.200	25.207	24.811	23.172	18.152	13.728

$$\begin{pmatrix}
4.034 & & & & & & & & & & \\
4.411 & 4.697 & & & & & & & & & \\
4.086 & 4.104 & 3.548 & & & & & & & & \\
4.037 & 4.038 & 3.703 & 3.886 & & & & & & & \\
4.318 & 4.808 & 4.251 & 4.799 & 6.566 & & & & & & \\
4.701 & 4.857 & 4.928 & 6.175 & 13.571 & -3.890 & & & & & \\
5.845 & 6.395 & 7.534 & 17.479 & -8.368 & -0.966 & -0.241 & & & & \\
7.982 & 9.988 & 20.793 & -11.146 & -1.973 & 0.074 & 0.496 & 1.049 & & & \\
11.758 & 25.122 & -11.195 & -1.549 & 0.387 & 1.347 & 1.695 & 2.404 & 3.694 & & \\
26.653 & -9.020 & 0.247 & 1.628 & 2.218 & 2.735 & 3.067 & 3.904 & 5.355 & 7.260 & \\
-7.652 & 1.472 & 3.109 & 3.263 & 3.405 & 3.740 & 4.077 & 5.015 & 6.630 & 8.825 & 11.759
\end{pmatrix}$$

The equation (2.3) which gives the estimate of the parameter '*a*' as shown below

$$\hat{b} = \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n} Z_{ij}$$

= 4.120030303

Equation (2.4) gives the estimates of the parameter '*b*'

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

= -29.90742398

Now, the normal equations for estimating the parameters *'a' and 'b'* are as follows

234.935 = 12 a + 144.131 b

2878.809504 = 144.131 a + 1746.108159 b

Hence, in this case

$\hat{a}_{(NE)}$ = -26.1971608

$\hat{b}_{(NE)}$ = 3.811122726

Result:

$\hat{a}_{(NE)}$ **= -26.1971608** $\hat{a}_{(stw)}$ **= -29.90742398**

$\hat{b}_{(NE)}$ **= 3.811122726** $\hat{b}_{(stw)}$ **= 4.120030303**

Estimated value of temperature ($\hat{y}$) by both the methods: Tezpur

Table: 3.2(a)

Length of Day (x)	**Observed Temperature (y)**	**Estimated Temperature** $\hat{y}_{(STW)}$	**Estimated Temperature** $\hat{y}_{(NE)}$	**Estimates of Error** $\lvert\hat{e}_{(STW)}\rvert = \lvert(y - \hat{y}_{(STW)})\rvert$	**Estimates of Error** $\lvert\hat{e}_{(NE)}\rvert = \lvert(y - \hat{y}_{(NE)})\rvert$
10.553	11.424	13.57125581	14.02161733	2.14725581	2.59761733
11.105	13.651	15.84551254	16.12536707	2.19451254	2.47435707
11.834	17.075	18.84901463	18.90366554	1.77401463	1.82866554
12.610	19.828	22.04615814	21.86109677	2.21815814	2.03309677
13.266	22.377	24.74889802	24.36119328	2.37189802	1.98419328
13.605	24.603	26.14558830	25.65316389	1.54258830	1.05016389
13.469	25.132	25.58526417	25.1348512	0.45326417	0.0028512
12.921	25.264	23.32748757	23.04635594	1.93651243	2.21764406
12.191	24.498	20.31986545	20.26423635	4.17813455	4.23376365
11.424	21.665	17.15980220	17.34110522	4.50519780	4.32389478
10.755	16.808	14.40350193	14.79146412	2.40449807	2.01653588
10.398	12.610	12.93265111	13.4308933	0.32265111	0.8208933
Total = 144.131	**Total = 234.935**	**Total = 234.9349999 ≅ 234.935**	**Total = 234.935**	**Sum of Absolute deviation** $\sum\lvert\hat{e}_{(STW)}\rvert$ = **26.04868557**	**Sum of Absolute deviation** $\sum\lvert\hat{e}_{(NE)}\rvert$ = **25.58367675**

Absolute Mean Deviation ($\bar{e}_{(STW)}$ **) = 2.170723798**

Absolute Mean Deviation ($\bar{e}_{(NE)}$ **) = 2.131973063**

1. Test of significance for estimated temperature obtained by Stepwise Application of Principles of Least Squares (stw).

The null hypothesis to be tested is

H_0: There is no significant difference between the values of observed temperature and estimated temperature.

Under the null hypothesis H_0, the test statistic is

$$t = \frac{(\bar{y} - \hat{\bar{y}}_{(STW)})}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad \text{with } (n_1 + n_2 - 2) \text{ d.f.}$$

Where $$S^2 = \frac{1}{(n_1 + n_2 - 2)}\left[\sum_{i=1}^{12} (y_i - \bar{y})^2 + (\hat{y}_i - \hat{\bar{y}}_{(STW)})^2\right] \quad \text{and} \quad n_1 = n_2 = 11$$

$$\bar{y} = \hat{\bar{y}}_{(STW)} = 19.57791667$$

$$|t|_{cal} = 0 \quad \text{and} \quad t_{(tab,5\%,20df)} = 1.725$$

$$|t|_{cal} < t_{(tab,5\%,20df)}$$

Thus, the null hypothesis H_0 is accepted.

Accordingly, it can be concluded that the difference between observed temperature and the corresponding estimated temperature is insignificant.

2. Test of significance for estimated temperature obtained by solution of normal equations (NE).

The null hypothesis to be tested is

H_0: There is no significant difference between the values of observed temperature and estimated temperature.

Under the null hypothesis H_0, the test statistic is

$$t = \frac{(\bar{y} - \hat{\bar{y}}_{(NE)})}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad \text{with } (n_1 + n_2 - 2) \text{ d.f.}$$

Where $$S^2 = \frac{1}{(n_1 + n_2 - 2)}\left[\sum_{i=1}^{12} (y_i - \bar{y})^2 + (\hat{y}_i - \hat{\bar{y}}_{(NE)})^2\right] \quad \text{and} \quad n_1 = n_2 = 11$$

$$S = 4.797$$

Since $$\bar{y} = \hat{\bar{y}}_{(NE)} = 19.57791667$$

$$|t|_{cal} = 0 \quad \text{and} \quad t_{(tab,5\%,20df)} = 1.725$$

$$|t|_{cal} < t_{(tab,5\%,20df)}$$

Thus, the null hypothesis H_0 is accepted.

Accordingly, it can be concluded that the difference between observed temperature and the corresponding estimated temperature is insignificant.

4. CONCLUSION:

The method, developed here, is based on the principle of the elimination of parameters first and then the minimization of the sum of squares of the errors whiles the ordinary least squares is based on the principle of the minimization of the sum of squares of the errors first and then elimination of parameters.

It is to be noted that the number of steps of computations in estimating the parameters by the method introduced here is less than the number of steps in estimation of parameters by the solutions of the normal equations. This implies that the error that occurs due to approximation in computation is less in the former than in the later.

We, therefore, may conclude that stepwise application of principles of least squares method is a simpler method of obtaining least square estimates of parameters of linear curve than the method of solving the normal equations. The following tables **(Table-4.1(a)) & (Table – 4.2(a))** show the values of *t* for the testing the significance of difference between the observed temperature and estimated temperature by both the method and comparison of their't' values

Table – 4.1(a)

Ex. No.	*Values of 't' in case of method of STW*	*Hypothesis*	*Significance /Insignificance*
1	$t_{cal} = 0 < t_{(tab,5\%,20df)} = 1.725$	H_0 Accepted	Insignificant
2	$t_{cal} = 0 < t_{(tab,5\%,20df)} = 1.725$	H_0 Accepted	Insignificant

Table – 4.2(a)

Ex. No.	*Values of 't' in case of method of solution of NE*	*Hypothesis*	*Significance /Insignificance*
1	$t_{cal} = 0 < t_{(tab,5\%,20df)} = 1.725$	H_0 Accepted	Insignificant
2	$t_{cal} = 0 < t_{(tab,5\%,20df)} = 1.725$	H_0 Accepted	Insignificant

The following table (Table-4.3(a)) shows the comparison between the two methods of estimating parameters.

Table – 4.3(a)

Ex. No.	*Comparison of 't' values of both the methods*
1	$t_{(STW)} = t_{(NE)}$
2	$t_{(STW)} = t_{(NE)}$

From the above table, it is found that both the method are almost equal in estimating parameters associated with a linear equation in case of unequal interval of the independent variable. In this study, attempt has been made for the case of linear curve only. Other types of the curves are yet to be dealt

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WIENER LOWER SUM OF COMPLETE K_N^R – GRAPH

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ABSTRACT :
The Wiener index of a graph is defined as the sum of distances between all pairs of vertices in a connected graph. Wiener index correlates well with many Physio Chemical properties of organic compounds and as such has been well studied over the last quarter of a century. In this paper we prove some general results on Wiener Lower and Upper Sum for graphs.
Keywords: ***K_n^r - Tree, Wiener Lower Sum of K_n^r – Tree***
MSC Code: 05CXX

1. INTRODUCTION

Chemical Graph theory is used to model physical properties of molecules called alkanes. Indices based on the graphical structure of the alkanes are defined and used to model both the boiling point and melting point of the molecules. Alkanes are organic compounds exclusively composed of carbon and hydrogen atoms. Molecular descriptors are "Terms that characterize a specific aspect of a molecule". Topological indices have been defined as those "Numerical values associated with chemical contribution for correlation of chemical structure with various physical properties, chemical reactivity or biological activity". A representation of an object giving information only about the number of elements composing it and their connectivity is named as topological representation of an object. A molecular graph is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. These points are named vertices and the lines are named edges in graph theory language.

Given the structure of an organic compound, the corresponding (molecular) graph is obtained by replacing the atoms by vertices and covalent bonds by edges (double and triple bonds also correspond to single edges unless specified otherwise).The Wiener index is one of the oldest molecular-graph based structure-descriptors, first proposed by the Chemist Harold Wiener as an aid to determining the boiling point of paraffins. The study of Wiener index is one of the current areas of research in Mathematical Chemistry. There are good correlations between Wiener index of molecular graphs and the Physico-Chemical properties of the underlying organic compounds.

2. COMPLETE GRAPH

A Complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. The complete graph with n vertices is denoted by K_n.

Let K_n be complete graph with n vertices, let $V(K_n) = \{u_1, u_2, u_3, \ldots u_n\}$ and $E(K_n) = \{u_iu_{i+1}/1 \leq i < n\}$ □ $\{u_nu_1\}$ □ $\{u_iu_j / 1 \leq i < n \text{ and } i < j \leq n\}$ be respectively vertex set and edge set of K_n.

Example - 1

Figure – (i) Complete K_9

Example – 2

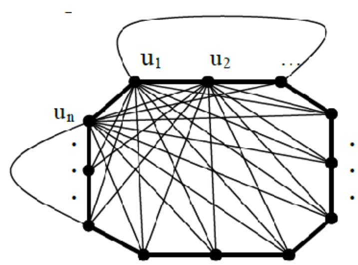

Figure – (ii) Complete graph K_n

Given K_n is a complete graph, let K_n^r graph be obtained by appending r pendant edges, $1 \leq r \leq n$ first edge to any one vertex, second edge to any remaining one vertex and third edge to any where remaining one vertex and so on

Let K_n^r be a complete graph with n vertices and r pendent vertices, let $V(K_n^r) = \{u_1, u_2, . . . u_n, v_1, v_2, v_3, . . .v_r\}$ and $E(K_n) = \{u_iu_{i+1}/1 \leq i < n\} \cup \{u_nu_1\} \cup \{u_i u_j / 1 \leq i < n \text{ and } i < j \leq n\} \cup \{u_pv_1, u_qv_2, . . . u_kv_r /1 \leq r \leq n \ \& \ 1 \leq p, q,. . . k \leq n\}$ be respectively the vertex set and the edge set of K_n^r.

Example – 3

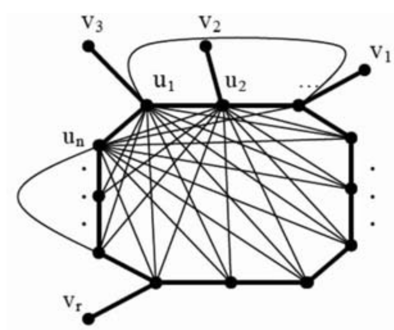

Figure – (iii) K_n^r

3. WIENER LOWER SUM $W^L(G)$

3.1 Definition

Let G = (V(G), E(G)) be a connected undirected graph, any two vertices u, v of V(G), □(u, v) denotes the minimum distance between u and v. Then the Wiener Lower Sum $W^L(G)$ of the graph is defined by

$$W^L(G) = \frac{1}{2}\sum_{u,v \in V(G)} \delta(u, v) \text{ where } \square(u, v) = \min d(u,v)$$

3.2 Theorem

The Wiener Lower Sum of a K_n^r – Graph is $W^L(K_n^r) = \dfrac{n(n-1) + 4r^2 + 2r(2n-1)}{2}$

Proof:

Let K_n be a complete graph with n vertices then the Wiener Lower Sum of K_n is

$$W^L(K_n) = \frac{n(n-1)}{2}.$$

Adding one pendant edge u_nv_1 to the end vertex then the Wiener Lower Sum is

$$W^L(K_n^1) = W^L(K_n) + \sum_i \delta(v_1, u_i),$$

where $1 \le i \le n$, $\delta(v_1,u_i)$ – minimum distance between vertices v_1 and u_i

$$W^L(K_n^1) = W^L(K_n) + 2(n-1) + 1$$

Again adding another pendant edge $u_{n-1}v_2$ to the last but one end vertex then the Wiener Lower Sum is

$$W^L(P_n^2) = W^L(P_n^1) + \sum_i \delta(v_1, u_i) + \sum_i \delta(v_2, u_i),$$

where $1 \le i \le n$, $\delta(v_2, u_i)$ – minimum distance between vertex v_2 to all the vertices of

$$K_n^1 \; W^L(K_n^2) = W^L(P_n^1) + 2(n-1) + 1 + 3$$

Also, again adding one more pendant edge $u_{n-2}v_3$ to the last but two end vertices then the Wiener Lower Sum is

$$W^L(K_n^3) = W^L(K_n^2) + \sum \delta(v_3, u_i) + \sum \delta(v_3, v_k)$$

where $1 \le i \le n$ and $1 \le k < 3$, $\delta(v_3,u_i)$ – Minimum distance between vertex v_3 to all the vertices of K_n^2

$$W^L(K_n^3) = W^L(K_n^2) + 2(n-1) + 1 + 3 + 3$$

Continue this process of adding pendent edge until $1 < r < n$ then the Wiener Lower Sum of K_n^r is

$$W^L(K_n^r) = W^L(K_n^{r-1}) + \sum \delta(v_r, u_i) + \sum \delta(v_r, v_k)$$

where $1 \le i \le n$, and $1 \le k < r$, Distance between vertex v_r to all the vertices of P_n^{r-1}

$W^L(K_n^r) = W^L(K_n^{r-1}) + 2(n-1) + 1 + \sum (2+3+\ldots r \text{ terms}) + \sum (3+4+\ldots r \text{ terms}) + \ldots$

$= \frac{n(n-1)}{2} + 2(n-1) + 1 + \ldots$ 'r' terms

$$W^L(K_n^r) = \frac{n(n-1) + 4r^2 + 2r(2n-1)}{2}$$

3.3 Programming in C

```
#include<stdio.h>
#include<conio.h>
void main()
  {
    int i,j,k,n,s=0,r;
    clrscr();
    printf("\nNo of Vertices in a graph - \t");
    scanf("%d",&n);
    printf("\nNo of Pendent Vertices - \t");
    scanf("%d",&r);
    for(i=1;i<n;i++)
         {
           for(j=i;j<n;j++)
             s=s+1;
         }
    printf("\nWiener Lower Sum is WL(Kn) = \t%d",s);
    for(i=1;i<=r;i++)
         {
           for(j=1;j<n;j++)
             s=s+2;
         }
```

```
        for(i=2;i<=r;i++)
            {
              for(j=1;j<i;j++)
                s=s+3;
            }
        s=s+r;
        printf("\nWiener Lower Sum is WL(Knr) = \t%d",s);
        getch();
      }
```

Output

No of Vertices in a graph - 8
No of Pendent Vertices - 5
Wiener Lower Sum is WL(Kn) = 28
Wiener Lower Sum is WL(Knr) = 133

4. CONCLUSION

Hence the Wiener Lower Sum of Knr - Tree are useful for systems like radar tracking, remote control, communication networks and radio-astronomy etc. Estimation of Wiener Lower Sum for other graph or tree is under investigation.

As it is well known that the valence of the carbon atom is four and vertex with more than degree four is not agreeable in reality, it is the interest of the chemist to use these work and study the stability of the compound and existence of the molecule with the properties with which we have analyzed the wiener index. We have taken only the concept of the wiener index and explored for general graphs with certain properties.

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DESIGNING A FUZZY INFERENCE RULE TO OPTIMIZE OVERALL PERFORMANCE OF VARIOUS DISK SCHEDULING POLICIES

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ABSTRACT :

Operating system provides an interface between user & Hardware. The Operating System manages hard disk to provide the fast access time. Most often, it has been observed disk scheduling algorithms incorporate Seek time as the only factor for optimal criteria & ignore the second factor i.e. Rotational Delay in the existing algorithms. In this paper both the factors seek time &rotational delay to schedule the disk have been considered with Equi-importance. The uncertainty associated with rotational delay has also been taken in account. Keeping in mind, a Fuzzy Inference system has been designed to optimize the overall performance of disk drive on the basis of If-then rule.

This paper is a combining effort of our previous work. Finally, we have presented comparative study with the Seek time & Disk access time for various scheduling algorithms in relative to our proposed algorithm with the Mamdani types Fuzzy Inference system.

Keywords: ***Disk Scheduling , Seek Time, Rotational Delay, Disk access time, Fuzzy inference rules, Linguistic Variables.***

1. INTRODUCTION: - A lot of Research work is available related to reduce the seek time, using different disk scheduling policies. But most of the policies are not providing any solution to handle the uncertainty & starvation problem associated with scheduling algorithms. For example: - suppose there are two requests with same distance. One request is close to the spindle & other towards the outer track. Our problem is to decide which request is to process first. The way to use fuzzy logic is to handle such kind of uncertainty.

In Recent years many researchers came forward with their idea to improve the disk performance. Hu. Weing [2005] proposed a new approach based on idea of disk arm & rotational position. Mohd. Talib [2009] explored the implementation of fuzzy disk scheduling algorithm using Fuzzy Inference system & IF Then Rules. Priya & Supriya [2014] extended the work and proposed a new approach to Fuzzy Disk Scheduling Algorithm. Sachin etal (2014) studied the effect on total transfer time using Zoned Bit Recording Technology. The disk scheduling techniques are applied on track request. Recently Ramanpreet kaur etal. [2015] came forward with two different ideas, one is to find the effectiveness of Disk Scheduling algorithm on changing Head position & Distance between Track Request, secondly to access the disk scheduling policy with Fuzzy Rotational Delay. Further Ramanpreet kaur etal. explored a new algorithm & found encouraging results.

This paper combines both the study in one single aspect. The Linguistic variables have been introduced on the basis of fuzzy reasoning consists of IF Then Rules. The New Proposed algorithm show the effective result compare to any other scheduling policy. The Fuzzy Triangular as well as Trapezoidal Function has been discussed. The paper is wider, relevant & more practical in real world environment. The overview of the results have been shown through Mamdani types structure as well as with Bar Diagram.

2. FUZZY RULE BASED GENERATION: - Fuzzy systems are a class of system belonging toknowledgebased systems.In this class, the knowledge is represented in the form of a Rule Based of a system. The process of designing involves following steps: -

1. Identify Input & output variables.
2. For these variables generate membership functions & decide their shapes such as triangular, trapezoidal, Z-type, S-type.
3. Generate Rule based for the system.
4. Select the type of Inference.
5. Select the type of aggregation.
6. Decide all the defuzzification technique & generate a crisp control action.

In this Paper we have applied **IF-THEN** Rules which converts fuzzy input into fuzzy output to transform the fuzzy output values into crisp values. We applied the triangular & trapezoidal arithmetic functions. Finally Mamdani type Fuzzy Inference system structure have been framed.

3. PROBLEM STRUCTURE: -In order to provide an efficient disk scheduling algorithm, we are using fuzzy logic to map two inputs to a single output. The input parameters are **ALGORITHM &ROTATIONAL DELAY&** the output parameter is **DISK ACCESS TIME.**

3.1 FORMULAE USED-

1) TOTAL SEEK TIME= $\sum_{i=1}^{n}$ **Seek Time (i)**

2)TRANSFER TIME= $\frac{\text{Total amount of data to be transferred}}{\text{Data Transfer Rate}}$

3) DISK ACCESS TIME= Seek Time+ Rotational Delay + Transfer Time

METHDOLOGY ADOPTED:

PARAMETERS	SPECIFICATIONS OF HARD DISK USED
Model	WDC WD 5000 LPVX-75V0TT0 -ATA DEVICE
Rotations Per minute	5400
Number of Track Request	8
List of Requests	{16, 75, 24, 21, 30, 80, 116, 63 }
Initial Head Position	66
Data Transfer Rate	147MB/s
Data to be transferred	20 MB
Transfer Time	136.05 m sec
Rotational Delay in Triangular fuzzy	8.3025 m sec
Rotational Delay in Trapezoidal fuzzy	6.90 m sec

4. DESIGNED FIS: - The Designed system consisting of two inputs namely, **Algorithm & Rotational Delay** & one output named **DISK Access Time**.

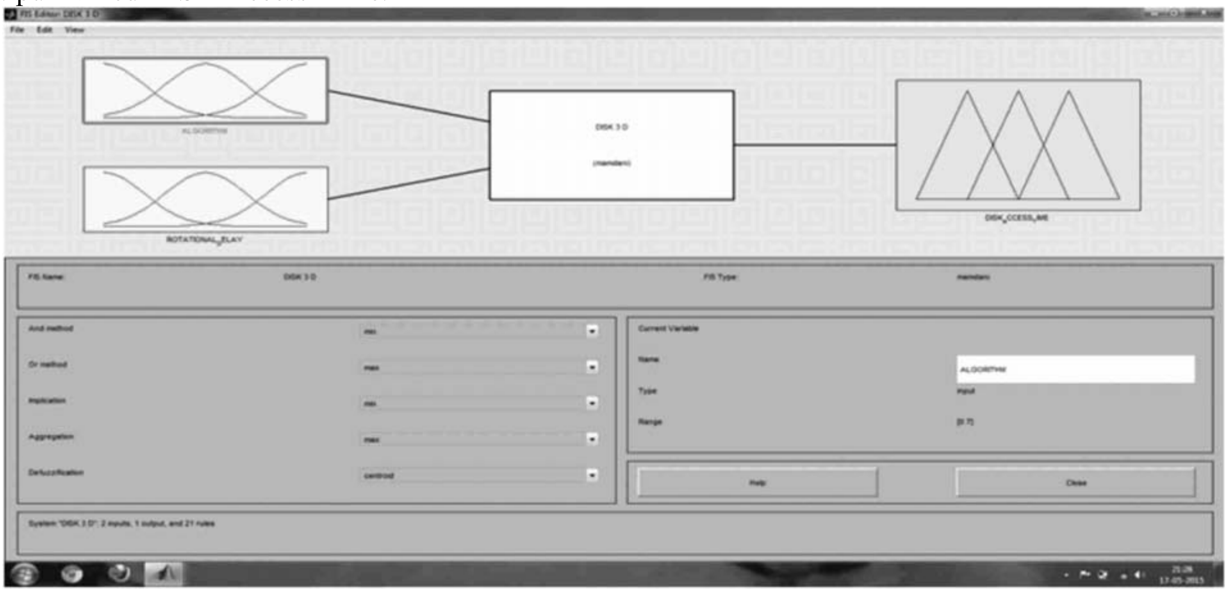


FIGURE 1.1: FIS FOR DISKSCHEDULING (TRIANGULAR FUZZY)

In this paper, Range is defined for different scheduling algorithms, such as:-

ALGORITHM	RANGE
FCFS	[0 1]
SSTF	[1 2]
SCAN	[2 3]
C-SCAN	[3 4]
LOOK	[4 5]
C-LOOK	[5 6]
NEW ALGORITHM-3	[6 7]

Rotational Delay defines under both the Membership function such as: -

1) Triangular Arithmetic Function
2) Trapezoidal Arithmetic Function

Based on the set of rule the values in the input parameters are mapped to output parameters. In our designed FIS, the output is optimized Disk Access Time that depends on the above defined inputs.

1) Triangular Arithmetic Function : -Rotational Delay of our disk in Triangular arithmetic form is calculated in fuzzy numbers **(5.5 m sec, 8.3 m sec, 11.11 msec)**.Here, the Triangular type membership function for all three linguistic variables are used such as **LOW, AVG, HIGH.**If Fuzzy set Ă=(a,b,c) then Yager's New defuzzification Formula is approximated as **(a+2b+c) / 4**

2) Trapezoidal Arithmetic Function : -Rotational Delay of our disk in trapezoidal arithmetic form is calculated in fuzzy numbers **(2.7 m sec, 5.5 m sec, 8.3 msec, 11.11 msec)**.Here the Trapezoidal type membership functions for all four linguistic variables are used such as **LOW,AVG,MORE, HIGH.**If Fuzzy set Ă=(a,b,c,d) then Chen's defuzzification Formula on the basis of functional principle is approximated as **(a+2b+2c+d) / 6**

4.1 TRIANGULAR FUZZY: -

DESCRIPTION: -

ALGORITHM	SEEK TIME	DISK ACCESS TIME
FCFS	311 m sec	455.35 m sec
SSTF	156 m sec	300.35 m sec
SCAN	182 m sec	326.35m sec
C-SCAN	223 m sec	367.35m sec
LOOK	150 m sec	294.35m sec
C-LOOK	197 m sec	341.35m sec
NEW ALGORITHM-3	150 m sec	294.35m sec

FIGURE 1.2: FUZZY INPUT PARAMETERS FOR DIFFERENT ALGORITHMS

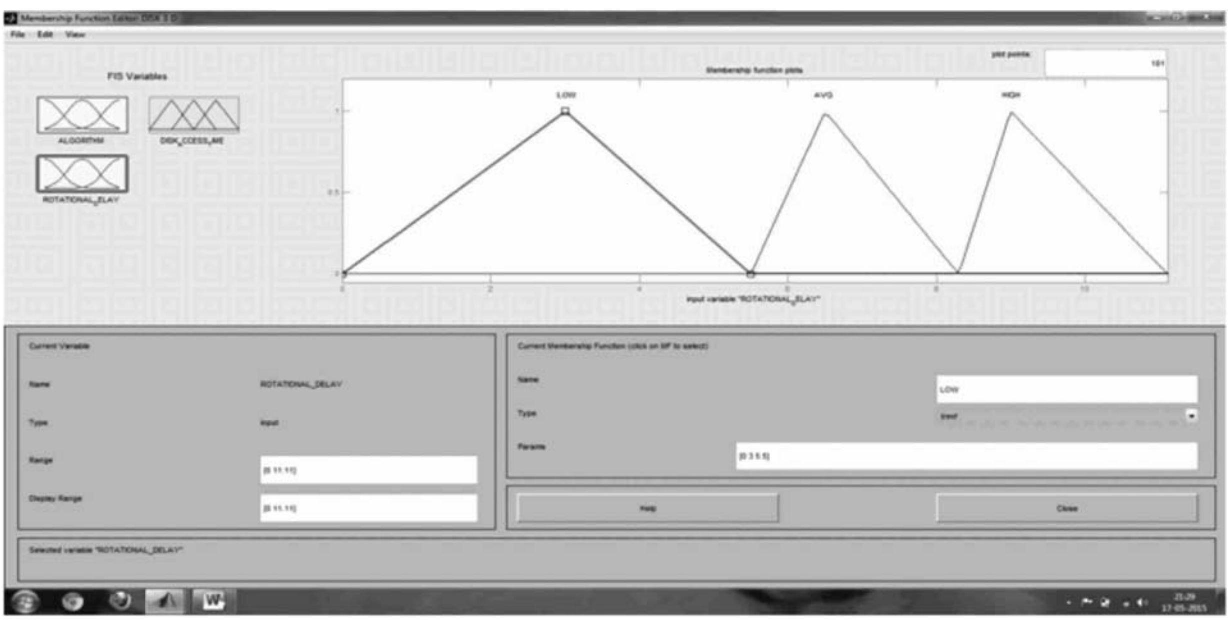


FIGURE 1.3: FUZZY INPUT PARAMETERS FOR ROTATIONAL DELAY

FIGURE 1.4:FUZZY OUTPUT PARAMETERS FOR DISK ACCESS TIME

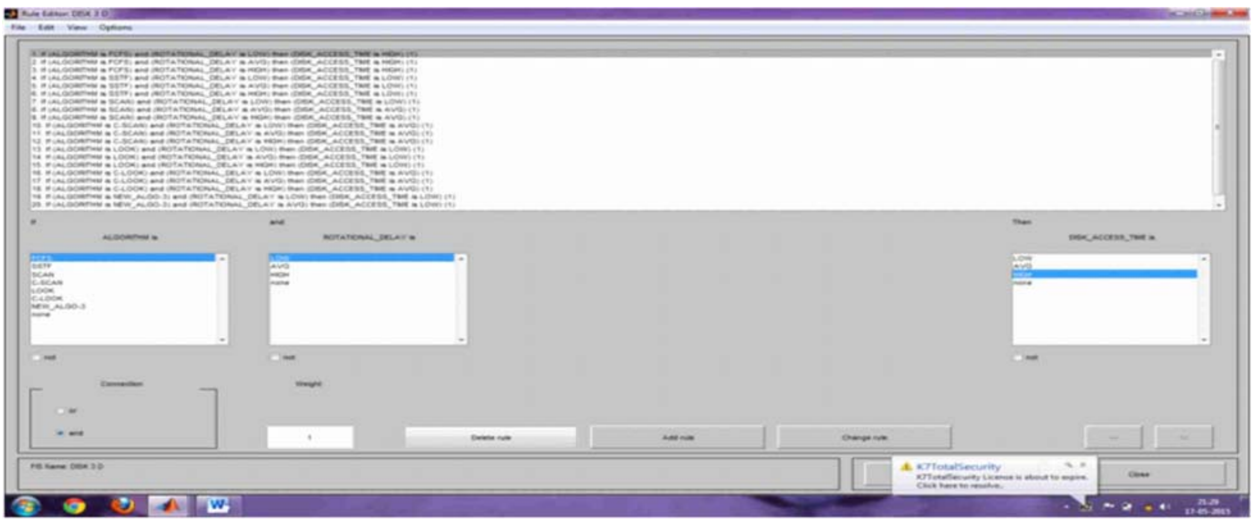


FIGURE 1.5: SET OF IF-THEN ELSE RULES

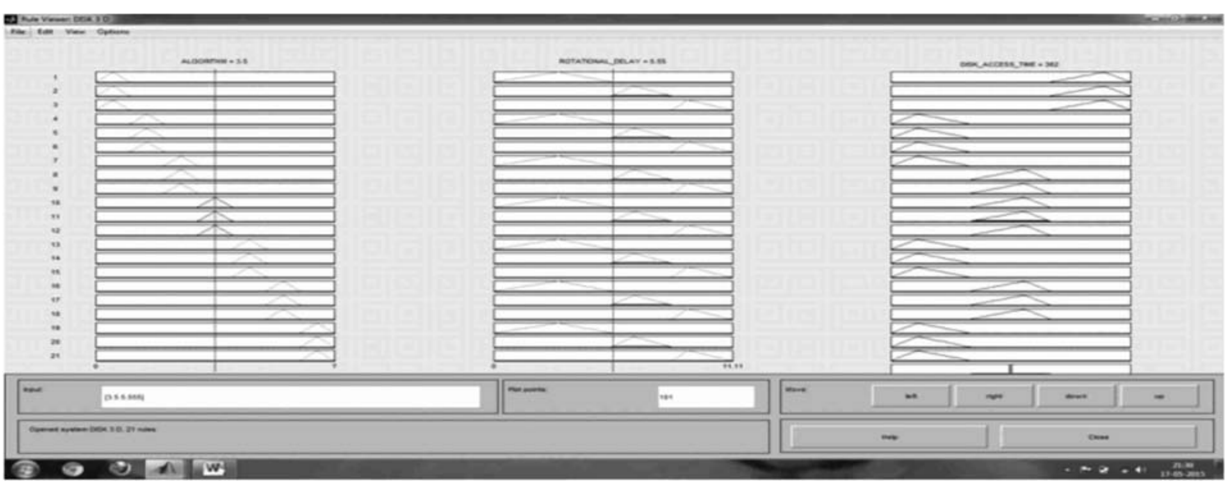


FIGURE 1.6: FUZZY INFERENCE RULES FOR ALGORITHM & ROTATIONAL DELAY V/S DISK ACCESS TIME

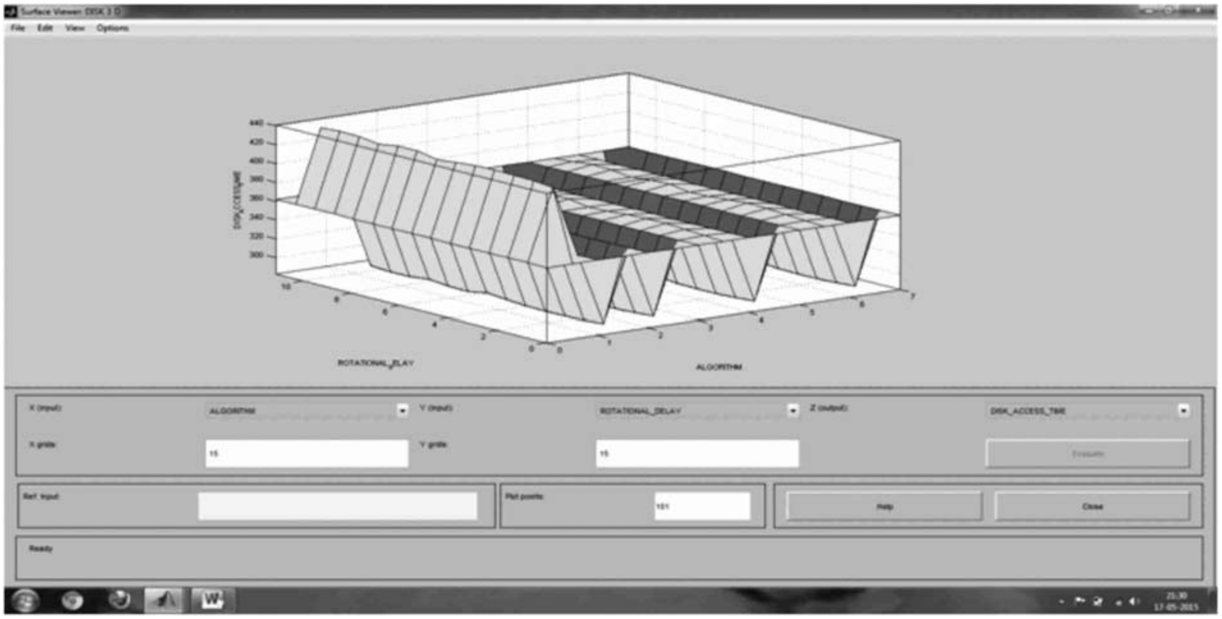


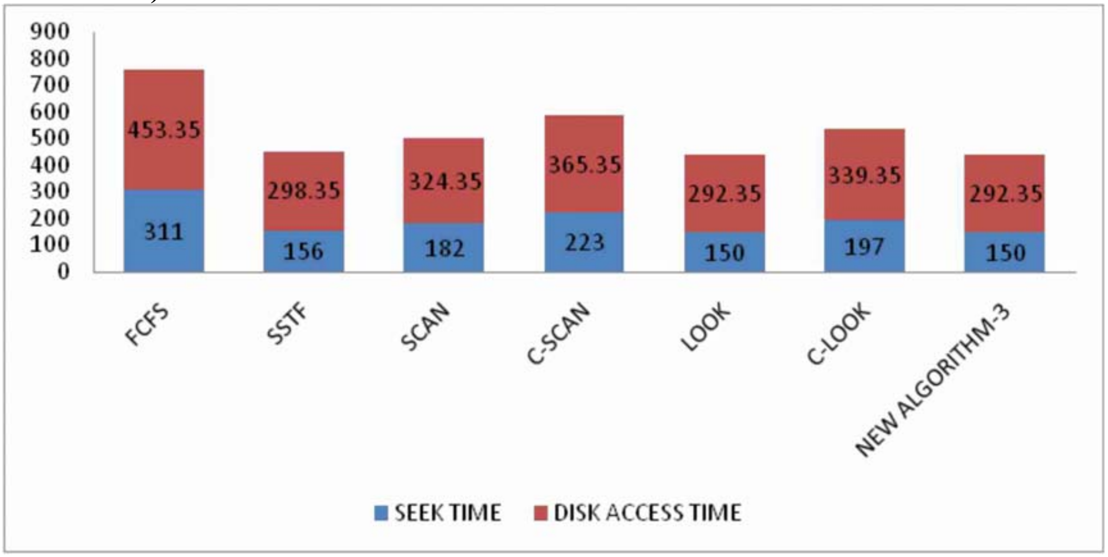
FIGURE 1.7: FUZZY SURFACE VIEW FOR ALGORITHM & ROTATIONAL DELAY V/S DISK ACCESS TIME

4.2 COMPARATIVE RESULTS OF DISK ACCESS TIME USING TRIANGULAR FUNCTION IN ROTATIONAL DELAY IN DIFFERENT SCHEDULING ALGORITHMS: -

ALGORITHM	ROTATIONAL DELAY	DISK ACCESS TIME
FCFS	**LOW**	**HIGH**
	AVG	**HIGH**
	HIGH	**HIGH**
SSTF	**LOW**	**LOW**
	AVG	**LOW**
	HIGH	**LOW**
SCAN	**LOW**	**LOW**
	AVG	**AVG**
	HIGH	**AVG**
C-SCAN	**LOW**	**AVG**
	AVG	**AVG**
	HIGH	**AVG**
LOOK	**LOW**	**LOW**
	AVG	**LOW**
	HIGH	**LOW**
C-LOOK	**LOW**	**AVG**
	AVG	**AVG**
	HIGH	**AVG**
NEW ALGORITHM-3	**LOW**	**LOW**
	AVG	**LOW**
	HIGH	**LOW**

In short, we can easily say that our**NEW ALGORITHM-3** produces good & effective result as compare to any other algorithm. And it will take less time in calculating **Seek Time**&**Disk Access Time** comparatively under taking triangular function in Rotational Delay in Fuzzy.

4.3 Comparative Graph among proposed and existing algorithm showing Seek time & Disk Access Time (in triangular function)



4.4 TRAPEZOIDAL FUZZY: -
DESCRIPTION: -

ALGORITHM	SEEK TIME	DISK ACCESS TIME
FCFS	311 m sec	453.95 m sec
SSTF	156 m sec	298.95 m sec
SCAN	182 m sec	324.95m sec
C-SCAN	223 m sec	365.95m sec
LOOK	150 m sec	292.95m sec
C-LOOK	197 m sec	339.95m sec
NEW ALGORITHM-3	150 m sec	292.95m sec

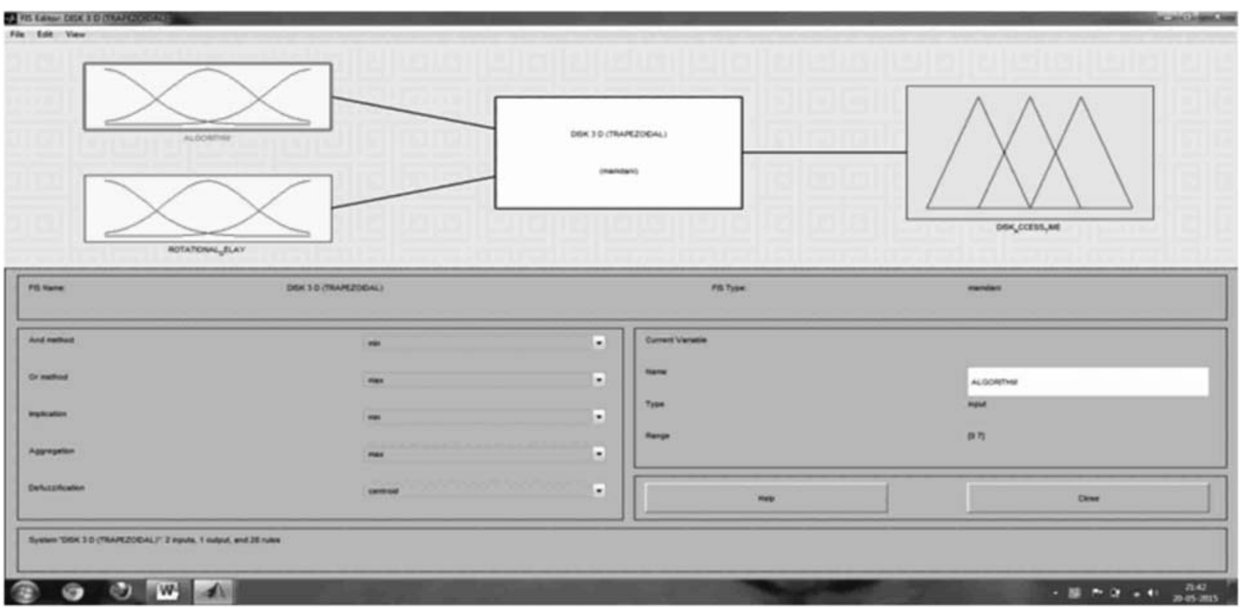


FIGURE 1.8: FIS FOR DISKSCHEDULING (TRAPEZOIDAL FUZZY)

FIGURE 1.9: FUZZY INPUT PARAMETERS FOR DIFFERENT ALGORITHMS

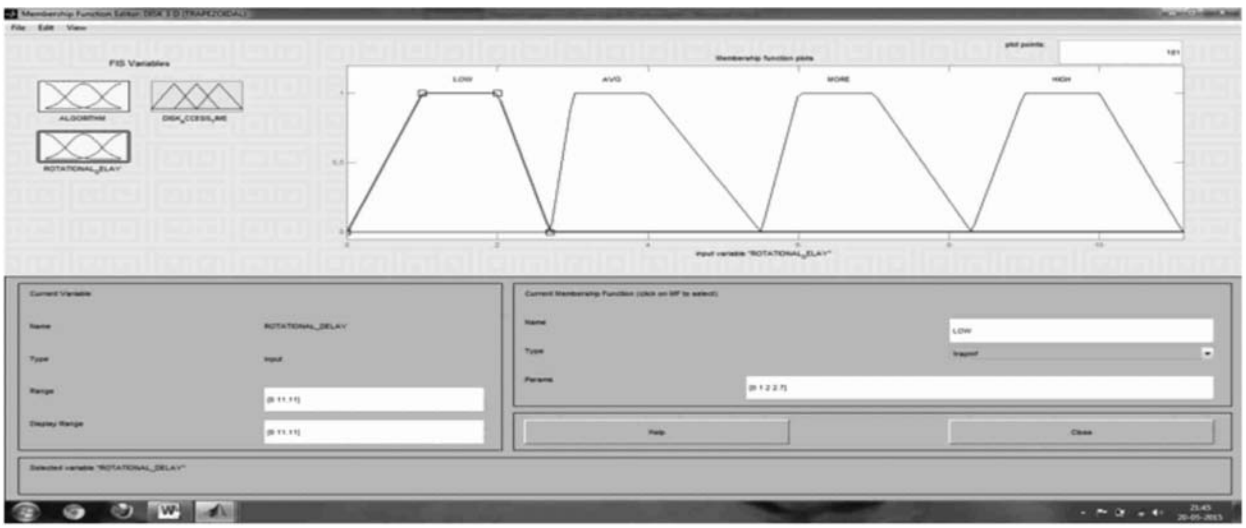


FIGURE 1.10: FUZZY INPUT PARAMETERS FOR ROTATIONAL DELAY

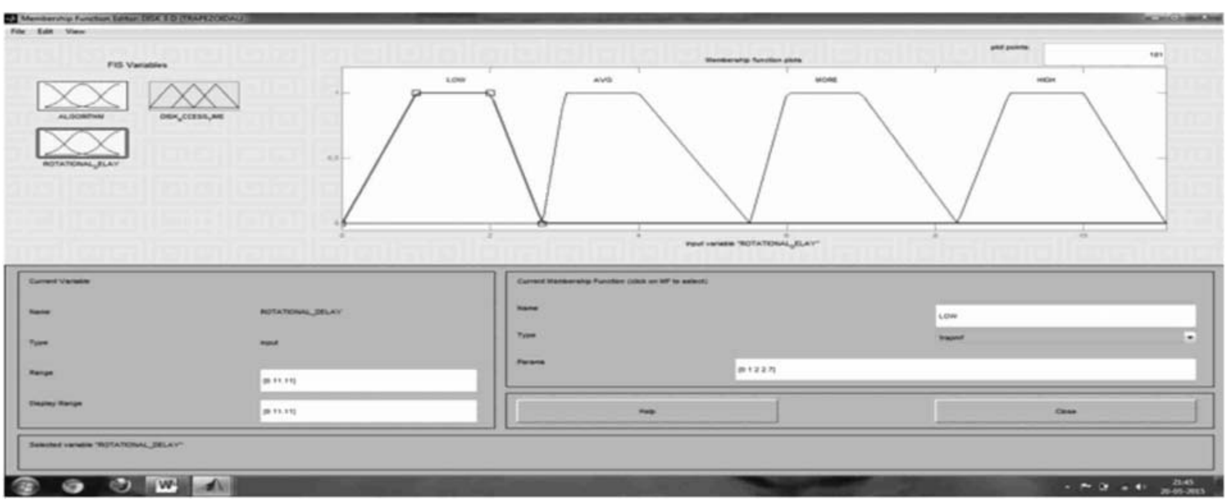


FIGURE 1.11: FUZZY OUTPUT PARAMETERS FOR DISK ACCESS TIME

FIGURE 1.12: SET OF IF-THEN ELSE RULES

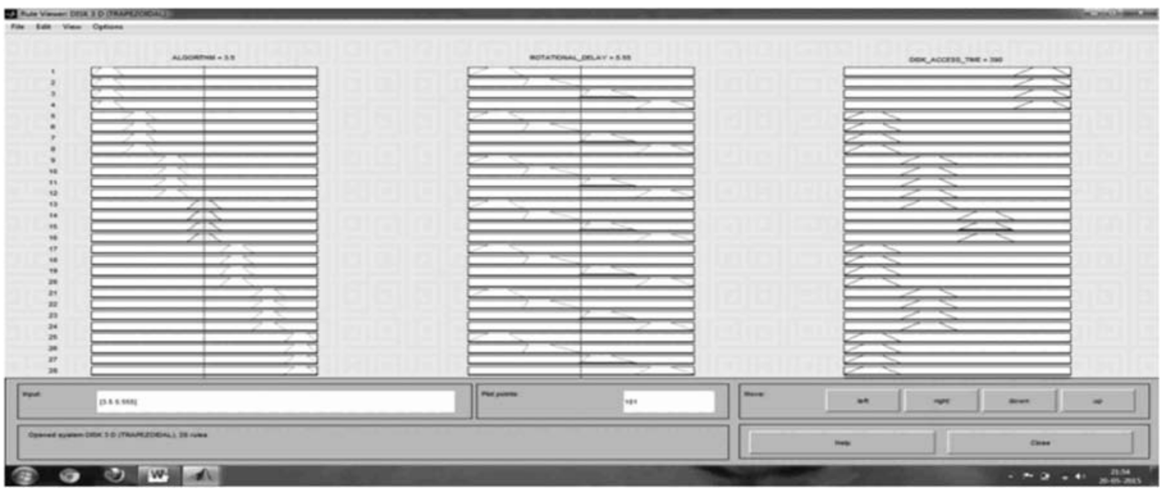


FIGURE 1.13: FUZZY INFERENCE RULES FOR ALGORITHM & ROTATIONAL DELAY V/S DISK ACCESS TIME

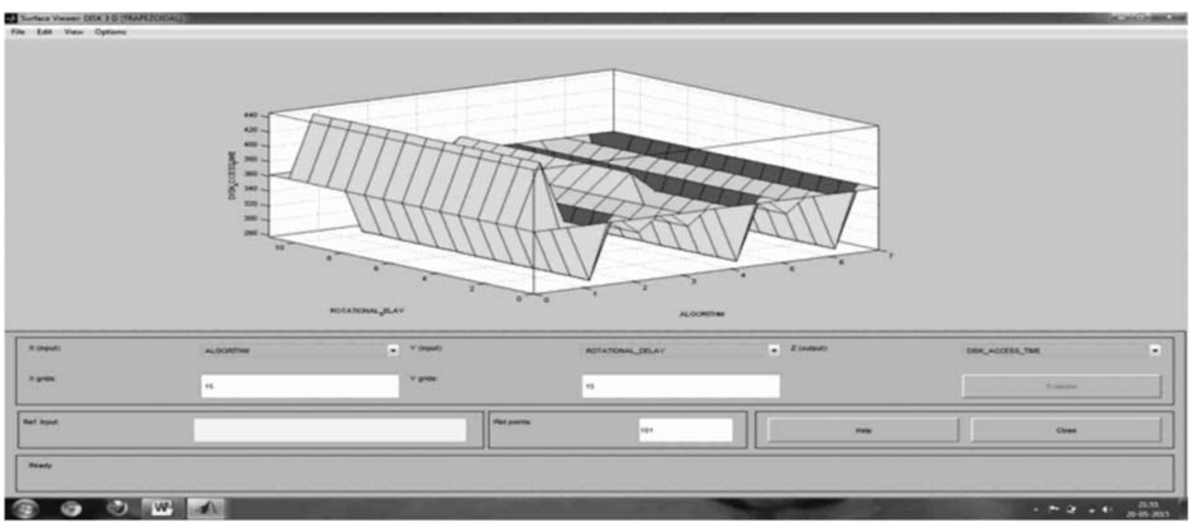


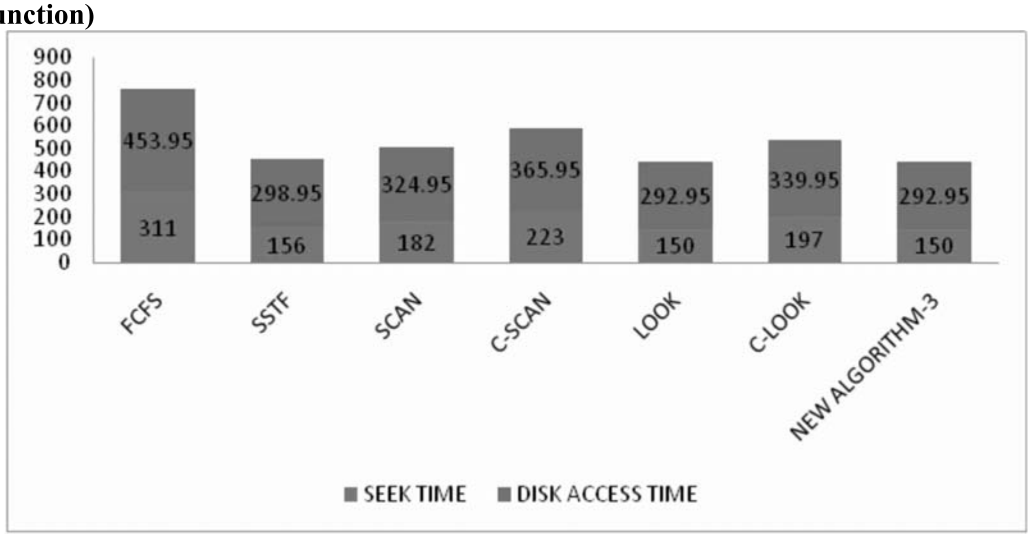
FIGURE 1.14: FUZZY SURFACE VIEW FOR ALGORITHM & ROTATIONAL DELAY V/S DISK ACCESS TIME

4.5 COMPARATIVE RESULTS OF DISK ACCESS TIME USING TRAPEZOIDAL FUNCTION IN ROTATIONAL DELAY IN DIFFERENT SCHEDULING ALGORITHMS: -

ALGORITHM	ROTATIONAL DELAY	DISK ACCESS TIME
FCFS	LOW	HIGH
	AVG	HIGH
	MORE	HIGH
	HIGH	HIGH
SSTF	LOW	LOW
	AVG	LOW
	MORE	LOW
	HIGH	LOW
SCAN	LOW	AVG
	AVG	AVG
	MORE	AVG
	HIGH	AVG
C-SCAN	LOW	AVG
	AVG	MORE
	MORE	MORE
	HIGH	MORE
LOOK	LOW	LOW
	AVG	LOW
	MORE	LOW
	HIGH	LOW
C-LOOK	LOW	AVG
	AVG	AVG
	MORE	AVG
	HIGH	AVG
NEW ALGORITHM-3	LOW	LOW
	AVG	LOW
	MORE	LOW
	HIGH	LOW

In short, we can easily say that our**NEW ALGORITHM-3** produces good & effective result as compare to any other algorithm. And it will take less time in calculating **Seek Time**&**Disk Access Time** comparativelyunder taking trapezoidal function in Rotational Delay in Fuzzy.

4.6 Comparative Graph among proposed and existing algorithm showing Seek time & Disk Access Time (in trapezoidal function)

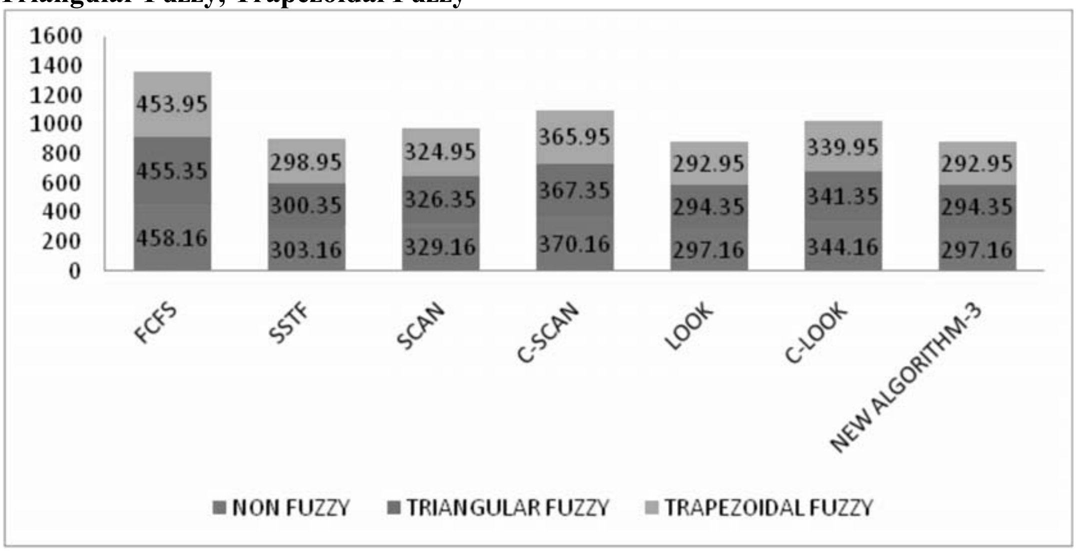


5. ANALYTICAL TABLE

Among proposed and existing algorithm showing Disk access time in Non-fuzzy, Triangular Fuzzy, Trapezoidal Fuzzy

ALGORITHM	NON FUZZY (in m sec)	TRIANGULAR FUZZY (in m sec)	TRAPEZOIDAL FUZZY (in m sec)
FCFS	458.16	455.35	453.95
SSTF	303.16	300.35	298.95
SCAN	329.16	326.35	324.95
C-SCAN	370.16	367.35	365.95
LOOK	297.16	294.35	292.95
C-LOOK	344.16	341.35	339.95
NEW ALGORITHM-3	297.16	294.35	292.95

5.1 Analysis through Graph among proposed and existing algorithm showing Disk access time in Non-fuzzy, Triangular Fuzzy, Trapezoidal Fuzzy



5. CONCLUSION: -

From the observed results, fuzzy surface view for rotational delay v/s disk access time and through bar diagram it is clear that the **NEW ALGORITHM-3** shows better performance than other disk scheduling algorithms (FIFO,SSTF,SCAN,C-SCAN and LOOK,C-LOOK). In our future problem **NEW ALGORITHM-3** can be better implemented in real time systems. In this paper we have shown which algorithm shows better result while calculating seek time & Disk Access time considering rotational delay in fuzzy. So we can easily say that algorithm 3 is applicable in any case. At last we show comparison between fuzzy and non-fuzzy data in all algorithms.

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STRUCTURAL TIME SERIES MODEL FOR FORECASTING POTATO PRODUCTION

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ABSTRACT :

The principal root and tuber crops of the tropics are potato (Solanum spp.) and grown almost in all states of India. Major potato growing states are Himachal Pradesh, Punjab, Uttar Pradesh, Madhya Pradesh, Gujarat, Maharashtra, Karnataka, West Bengal, Bihar and Assam. UP, West Bengal, Bihar and Punjab together account for about 86% of India's production. A univariate structural time series model based on the traditional decomposition into trend, seasonal and irregular components is defined. Purpose of present paper is to discuss STM methodology utilized for modelling time-series data in the present of trend, seasonal and cyclic fluctuations. Structural time series model are formulated in such a way that their components are stochastic, i.e. they are regard as being driven by random disturbances. A number of methods of computing maximum Likelihood estimators are then considered. These include direct maximization of various times domain likelihood function. Once a model is estimated, its suitability can be assessed using goodness fit statistics and model used to forecast for five leading years. In our study the model developed for potato production, from the forecasting available.

Keywords: ***Structural time series model, forecast, Kalman filter, goodness of fit***

INTRODUCTION

The principal root and tuber crops of the tropics are potato (*Solanum* spp.), cassava (*Manihot esculenta* Crantz), yam (*Dioscorea* spp.), sweet potato (*Ipomoea batatas* L.), and edible aroids (*Colocasia* spp. and *Xanthosoma sagittifolium*). They are widely grown and consumed as subsistence staples in many parts of Africa, Latin America, the Pacific Islands and Asia. Root and tuber crops are second only in importance to cereals as a global source of carbohydrates. They also provide some minerals and essential vitamins, although a proportion of the minerals and vitamins may be lost during processing. Potato is grown almost in all states of India. Major potato growing states are Himachal Pradesh, Punjab, Uttar Pradesh, Madhya Pradesh, Gujarat, Maharashtra, Karnataka, West Bengal, Bihar and Assam. UP, West Bengal, Bihar and Punjab together account for about 86% of India's production. However, potato consumption per capita in India (14.8 qt/head/year) is one of the lowest in the world and hardly 1% of the potato is processed. Potato is the world's fourth important food crop after wheat, rice and maize because of its higher yield potential along with high nutritive value. Potato is an economically important staple crop in both developed and developing countries. India ranks 4^{th} in terms of area and is the 2^{nd} largest country in world in production of potato after China.

ARIMA time series methodology is widely used for modelling time-series data. This methodology can be applied only when either the series under consideration is stationary or it can be made so by differencing, de-trending, or by any other means. Another disadvantage is that this approach is empirical in nature and does not provide any insight into the underlying mechanism. An alternative mechanistic approach, which is quite promising, is the Structural Time Series Modelling (Harvey, 1996). Purpose of present paper is to discuss STM methodology utilized for modelling time-series data in the present of trend, seasonal and cyclic fluctuations. Structural time series model are formulated in such a way that their components are stochastic, i.e. they are regard as being driven by random disturbances. Forecasts are made by extrapolating these components into the future. Harvey and Todd (1983) compare the forecast made by a basic form of the structural model with the forecast made by ARIMA models and conclude that there may be strong arguments in favour of using structural models in practice. Structural models are applicable in the same situations where Box-Jenkins ARIMA models are applicable; however, the structural models tend to be more informative about the underlying stochastic structure of the series. In another paper Harvey (1996) showed structural models can be used to model cycle in macro economics time series. The forecast obtained from particular model depends on certain variance parameter. In another paper Ravichandran, Prajneshu (2002) applied the STM models for cyclical fluctuation in all-India food grain production. The modeling and forecasting using these study found out to be better than traditional ARIMA forecasting model and other studies for forecasting India's rice productions Ravichandran and Muthuraman (2006).

The key to handling structural time-series models is the state space form, with the state of the system representing the various unobserved components, such as trend, cyclical or seasonal fluctuations. Once in state space form (SSF), the Kalman filter provides the means of updating the state, as new observations become available. Once a model is estimated, its suitability can be assessed using goodness fit statistics. Potato area (million ha) production (million tonnes) and yield (qt/ha) in India data for the period of 1950-51 to 2013-2014 were analyzed by structural time series model used to forecast for five leading years.

MATERIALS AND METHODS

The study mainly confined to secondary data of area, production and yield on Potato tuber crop of 64 year was collected for period 1950-51 to 2013-14. Data collected from various publications, Government of India were subjected to analyze through structural time series model. The data are analyzed by using software like MS-EXCEL and Statistical Analysis System (SAS). Structural time series model was adopted to observe the forecast model, the model used was:

Structural time Series Model for trend: A structural time series model is set up in term of its various components, like trend, cyclic fluctuations and seasonal variation, i.e.

$$Y_t = T_t + C_t + S_t + \varepsilon_t \qquad (1)$$

Where Y_t is the observed time-series at time t, T_t, C_t, S_t, ε_t are the trend, cyclical, seasonal and irregular components.

(i) ***Local Level Model** (LLM):* In the absence of seasonal and cyclical components, eq. (1) reduce to

$$Y_t = \mu_t + \varepsilon_t,\ \varepsilon_{t\,t} \sim N(0, \sigma_\varepsilon^2), \qquad t = 1, 2... T \qquad (2)$$

Where $\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t$ and $\beta t = \beta_{t-1} + \varepsilon_t$

(ii) **Goodness of fit:** Goodness of fit statistics is used for assessing over all models fit. Basic measure of goodness of fit in time series model is prediction error variance. Comparison of fit between different models is based on Akaike information criterion (AIC).

$$AIC = -2 \log L + 2n,$$

Where L is the likelihood function, which is expressed in term of estimated one-step-ahead prediction errors $\hat{u}_t = Y_t - \hat{Y}_t/_{t-1}$. Here n is the number of hyper parameters estimated from the model. Schwartz-Bayesian information criterion (BIC) is also used as a measure of goodness of fit which is given as

$$BIC = -2\log L + n \log T,$$

Where T is total number of observations. Lower the value of these statistics better is the fitted model.

RESULTS AND DISCUSSION

Forecasting with structural time-series model: Structural time series model are developed basically to forecast the corresponding variable. To judges the forecasting ability of the fitted model important measure of the sample period forecasts accuracy was computed. The AIC for potato tuber crop area, production and yield to be obtained -167.8, 264.56 and 1056.3 and BIC to be obtained -161.5, 270.89 and 1062.6 respectively. Potato area forecast for the year 2019 to be about 2.31 million hectare with upper and lower limit 2.03 and 2.59 million hectares, production to be about 60.47 million tonnes with upper and lower limit 49.67 and 71.26 million tonnes and yield about 256.92 qt/ha with upper and lower limit 236.52 and 277.32 qt/ha respectively. From the below table it is observed that the forecasts using Structural time series model shows an increasing trend for area, production and yield of potato crop in India. The validity of these forecasts can be checked when the actual data is available for the lead years. Forecasts for the future years from 2015 to 2019 by using the Structural time series models presented in the following table.

Table 1. Trend information

Name	Intercept	Slop	AIC	BIC
Area	1.98	0.054	-167.8	-161.5
Production	45.13	2.57	264.56	270.89
Yield	227.84	4.84	317.62	323.95

Table 2. Forecast for India's total potato area, production and yield

	Area (Million ha)		Production (Million Tonnes)		Yield (qt/ha)	
Year	Forecast	Standard Error	Forecast	Standard Error	Forecast	Standard Error
2015	2.09	0.075	50.24	2.60	237.53	4.68
2016	2.15	0.093	52.80	3.26	242.38	6.12
2017	2.20	0.110	55.36	3.97	247.22	7.54
2018	2.25	0.127	57.91	4.72	252.07	8.96
2019	2.31	0.144	60.47	5.51	256.92	10.40

Table 3. Fitting of Structural time series

Year	Area			Production			Yield		
	Lower Limit	Upper Limit	Model Width	Lower Limit	Upper Limit	Model Width	Lower Limit	Upper Limit	Model Width
2015	1.94	2.24	0.30	45.14	55.34	10.20	228.37	246.70	18.33
2016	1.96	2.33	0.37	46.40	59.20	12.80	230.37	254.40	24.03
2017	1.98	2.42	0.44	47.57	63.14	15.57	232.44	262.02	29.58
2018	2.00	2.50	0.50	48.66	67.16	18.5	234.50	269.65	35.15
2019	2.02	2.59	0.57	49.67	71.26	21.59	236.52	277.32	40.80
Average width			**0.43**	**Average width**		**15.73**	**Average width**		**29.57**

CONCLUSION

In our study the structural time-series model has been developed for potato area, production and yield, from the forecasting available. By using the developed model, it can be seen that forecasted areas, production and yield increases the next five years. The validity of the forecasted value can be checked when the data for the lead periods become available. The model can be used by researchers for forecasts potato areas, production and yield in India. However, it should be updated from time to time with incorporation of current data. From the above forecasts for the lead periods show that there is a small change in the forecasts of area, production and yield of potato in India.

There is a need to adopt the high yielding varieties and improved package of practices for increasing the production in India.

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AN INVENTORY MODEL OF PERISHABLE PRODUCTS WITH DISPLAYED STOCK LEVEL DEPENDENT DEMAND RATE

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ABSTRACT :
In this paper, an inventory model of perishable items with displayed stock level dependent demand rate has been developed. The Model has been discussed for three components demand rate of deteriorating items with respect to DSL in a showroom shop. The proposed model is developed under the realistic assumption that the storage capacity of the showroom shop is limited. Shortages are allowed and backlogged.

Keywords: ***Perishable, displayed stock level and shortages.***

1. INTRODUCTION:

There is great competition in the market now a days. It is observed that the glamorous display of an item in large quantity in modern light and electronic arrangement influences the customers. **Levin et al** (1972) presented a functional relationship between the demand and displayed stock level (DSL). **Silver** and **Peterson** (1985) observed that sales at retail shop trend to be proportional to DSL. This observation attracted the researchers to analyze the modeling aspects of this phenomenon. **Baker & Urban** (1988) first developed a basic model and noted that the demand rate would decline along with the stock level during the entire cycle. They assumed the power form of demand depending on DSL. **Mandal & Phaujdar** (1989) presented a production inventory model for deteriorating items assuming the consumption rate as a general function of on hand stock level during stock-in as well as stock-out period. **Dutta & Pal** (1990) modified the model of **Baker & Urban (**1988) by incorporating two component demand rate. **Pal et al** (1993), **Giri et al** (1996), **Padmanabhan & Vrat** (1995) and **Mandal & Phaujdar** (1989) extended this work. **Maity & Maity** (2007) developed a realistic multi-item production inventory model with shortages. **Paul et al** (1996) discussed a deterministic inventory model with two component demand with fully backlogged shortages. They assumed that the demand rate is stock dependent reducing to a certain limit and then a constant. Practically the DSL dependence of demand rate within a certain range and then it becomes constant. Such type of demand rate is known as three component demand rate. A completely backlogged shortage is an unrealistic assumption in practice. Some customers may wait for a long time to purchase goods due to some crucial factors such as genuine price of the item, discount facility, locality of the shop, goodwill of the retailer and prompt service etc. But some customers cannot wait due to their urgent need or due to some other reasons.

It is well known fact that advertisement of an item changes its demand. The selling price variation also affects its demand. **Ladany & Sterrlub** (1974) discussed the effect of price variation on demand. **Mahapatra & Maiti**

(2005) developed multi objective inventory system of stochastically deteriorating items in which demand was considered as a function of inventory level.

In this paper, an inventory model has been developed for three component demand rate of deteriorating items with respect to DSL in a showroom shop. The proposed model is developed under the realistic assumption that the storage capacity of the showroom shop is limited. Shortages are allowed and backlogged. According to the size of the showroom shop different cases have been discussed. The optimal solution of the system has been obtained. Numerical example has been given to illustrate the proposed model.

2. ASSUMPTIONS AND NOTATIONS:

The following assumptions have been used in developing the proposed model:

1. Replenishment rate is infinite with replenishment size finite.
2. The inventory planning horizon is infinite.
3. The inventory system has one item with one stocking point.
4. The order will be placed at the beginning of each cycle and the lot is delivered in one batch.
5. Lead time is zero
6. Deterioration rate θ is constant.
7. There is no quantity discount.
8. Shortages are allowed and partially back logged.

The following notations are used in the proposed model:

C_1: Inventory carrying cost per unit per unit time.

C_2: Shortages cost per unit per unit time.

C_3: Purchase cost per unit.

C_4: Ordering cost per order.

C_5: Deterioration cost per unit

p: Selling price per unit.

A: Frequency of advertisement per cycle.

G: Cost per advertisement.

T: Length of each cycle.

W: Storage capacity (in units) of showroom.

Q: Ordering quantity per cycle.

S: Highest stock level at the beginning of each cycle after fulfilling the backlogged quantity.

R: Stock level of re-order point.

$q(t)$: Inventory level at any time t.

3. MATHEMATICAL MODEL AND ANALYSIS:

Let the demand rate $d(A,p,q)$ be deterministic and function of marketing parameters *A*, the number of advertisements, *p* the selling price per unit and *q* is the current stock level in the showroom when *q* varies between S_0 and S_1 and beyond this range it becomes constant with respect to the displayed stock level. During stock out period, it is assumed that demand rate is δ (a constant) times the in stock level.

Thus we have

$$d(A,p,q) = f(A,p,S_1), \text{ for } q > S_1$$
$$= f(A,p,q),\ S_0 < q < S_1$$
$$= f(A,p,S_0),\ 0 \le q \le S_1$$
$$= \delta\ f(A,p,S_0),\ q < 0$$

where $0 < \delta \le 1$ and f is a function of *A*, *p* and *q* which may have different functional forms. Let deterioration rate be constant θ.The functional form of the stock level-dependent demand rate may be in different form such as power form (αq^{β}), the linear form $(\alpha + \beta q)$, the quadratic form $(\alpha + \beta q + \gamma q^2)$, exponential form $(\alpha e^{\beta q})$etc. with respect to the displayed stock level. Here we consider the linear form of the displayed inventory as given below:

$$f(A,p,q) = \alpha A^m p^{-e} q, \text{ where } \alpha > 0, e \ge 1, m \ge 0.$$

Let *S* be initial stock which will be less than or equal to W, the storage capacity of the showroom. There may be the following three cases:

Case 1: $S_0 \le W \le S_1$,

Case 2: $W \ge S_1$,

Case 3: $W \le S_0$.

The equal sign in the above inequality shows the smooth transition from one case to the other. The stock level depleted due to demand and deterioration. The third case may be rejected because in this case the consumption rate level never becomes DSL dependent. If R is the stock level of reorder then there may be the following sub cases of case 1 and case 2:

Case 1(i): $S_0 < R < S \le W < S_1$,

Case 1(ii): $0 \le R < S_0 < S < W < S_1$,

Case 1(iii): $S_0 \le S < W < S_1$ and $R < 0$,

Case 2(i): $S > S_1$ and $R > S_0$,

Case 2(ii): $S > S_1$ and $0 \le R \le S_0$,

Case 2(iii): $S > S_1$ and $R < 0$.

In the case 1(i) the demand rate never becomes constant. The inventory is replenished before reaching S_0. In case 1(ii), there is displayed stock level dependence of demand rate when stock level drops from S_1 to S_0. The selling rate becomes constant beyond the stock level S_0. In case 1(iii), the demand rate remains same in stock in as well as in stock out position. In case 2(i) the demand rate is constant between the stock level S and S_1. Then it depends on DSL. In case 2(ii), the demand rate is constant in the beginning, then it depends on DSL and finally again it becomes constant. In case 2(iii) all possible forms of demand rate exist and there is also the stock-out position.

Here the case 2(iii) has been discussed in detail.

Case 2(iii): $S > S_1$ and $R < 0$. In this case, the order quantity is given by

$Q = S + |R|$ and the total time period (cycle time) T is given by

$$T = \int_S^{S_1} \frac{dq}{f(A,p,S_1) - \theta q} + \int_{S_1}^{S_0} \frac{dq}{f(A,p,q) - \theta q} + \int_{S_0}^{0} \frac{dq}{f(A,p,S_0) - \theta q} - \frac{R}{\delta f(A,p,S_0)}$$

$$= \frac{1}{\theta} \log \frac{[f(A,p,S_1) - \theta S]}{[f(A,p,S_1) - \theta S_1]} \frac{[f(A,p,S_0) - \theta S_0]}{[f(A,p,S_0)]} - \frac{R}{\delta f(A,p,S_0)}. \qquad \ldots(1)$$

The total inventory carrying cost is given by

$$C_{CAR} = C_1 \left[\int_{S_1}^{S} \frac{qdq}{f(A,p,S_1) - \theta q} + \int_{S_0}^{S_1} \frac{qdq}{f(A,p,q) - \theta q} + \int_0^{S_0} \frac{qdq}{f(A,p,S_0) - \theta q} \right]. \qquad \ldots(2)$$

The total shortage cost is given by

$$C_{SHO} = C_2 \int_R^0 - \frac{qdq}{\delta f(A,p,S_0)}. \qquad \ldots(3)$$

The deterioration cost is given by

$$C_{DET} = C_5 \left[S - \left\{ \int_{S_1}^{S} \frac{dq}{f(A,p,S_1) - \theta q} + \int_{S_0}^{S_1} \frac{dq}{f(A,p,q) - \theta q} + \int_0^{S_0} \frac{dq}{f(A,p,S_0) - \theta q} \right\} \right]. \qquad \ldots(4)$$

Thus during the replenishment cycle, the total cost (TC) is given by

TC= Ordering cost + Inventory carrying cost + Inventory shortage cost + Deterioration cost + Purchase cost + Advertisement cost

$$= C_4 + C_3(S + |R|) + C_{CAR} + C_{SHO} + C_{DET} + C_{ADV}. \qquad \ldots(5)$$

Therefore, the net profit is the difference between the sales revenue and the total cost TC which is given by

$$X = (p - C_3)(S + |R|) - (C_4 + C_3(S + |R|) + C_{CAR} + C_{SHO} + C_{DET} + C_{ADV})$$

Hence the profit function (average profit per unit time) is given by

$$\overline{X} = \frac{X}{T}. \qquad \ldots(6)$$

Here the profit function is a function of two variables S and R. We have to find the optimal values of *S, R, Q* and *T* by maximizing the profit function $\overline{X}(S,R)$. The necessary conditions for $\overline{X}(S,R)$ to be maximum are that

$$\frac{\partial \overline{X}}{\partial S} = 0 \text{ and } \frac{\partial \overline{X}}{\partial R} = 0 . \qquad \ldots(7)$$

Equations (7) are, in general, nonlinear equations for S and R. Therefore the closed form solution of it can not be obtained for S and R. Only numerical solution can be derived by using a suitable numerical method.

It can be shown that the solution obtained from (7) will provide alternately negative and positive principal minors of the associated Hessian matrix.

Let $S^{\bullet}$ and $R^{\bullet}$ be the optimal values of *S* and *R*. In this analysis the capacity constraint $S \le W$ of the showroom shop has been ignored. If the capacity constraint is satisfied then the solution $S^{\bullet}$ and $R^{\bullet}$ is the required optimal solution.

5. CONCLUSIONS:

In this paper, an inventory model has been developed for three component demand rate of deteriorating items with respect to DSL in a showroom shop. The proposed model is developed under the realistic assumption that the storage capacity of the showroom shop is limited. Shortages are allowed and backlogged partially. This model can further be extended for variable deterioration rate and probabilistic demand rate. In future work the other cases of the proposed model may be discussed. Credit policy, discount facility and inflation factor can also be taken into consideration of further studies.

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TWO NEW BIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS AND THEIR PROPERTIES

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ABSTRACT :
Recently Kotoky and Chakarborty (2013) gave the mathematical expression for probability mass functions of bivariate Poisson distribution and its re-parameterization including some properties. On the basis of these expressions and properties an attempt has been made to derive two new bivariate negative Binomial distribution as a compound distribution of bivariate Poisson distribution. The different properties of new bivariate negative Binomial distribution have also been investigated.

Keywords: ***Bivariate Poisson distribution, Bivariate negative binomial distribution, Marginal distribution, Compound distribution.***

1. INTRODUCTION

Guldberg (1934) developed bivariate negative binomial distribution starting with bivariate Bernoulii distribution. Arbous and Kerrich (1951) derived the bivariate negative binomial distribution using compounding. Marshall and Olkin (1985) developed the modified version of the Guldberg (1934). In this article an attempt has been made to derive two new bivariate negative binomial distributions as compound distributions of bivariate Poisson distribution derived by Kotoky and Chakraborty (2013).

1.1 Bivariate Poisson (Kotoky and Chakraborty, 2013)

A random vector (Y_1, Y_2) follows the bivariate Poisson (Kotoky and Chakraborty, 2013) distribution parameters $(\lambda_1, \lambda_2, \beta, \alpha)$ if its pmf is given by

$$\Pr(Y_1 = y_1, Y_2 = y_2) = \exp(-\lambda_2)(\lambda_1\beta/(1+\alpha))^{y_1} \exp(-\lambda_1\beta/(1+\alpha)) \sum_{(j_1, j_2)\in T}((\alpha)^{j_1}((\lambda_2\beta - \alpha\lambda_1\beta/(1+\alpha))^{j_2}(\lambda_2(1-\beta))^{y_2-j_1-j_2})/j_1!\,j_2!(y_2-j_1-j_2)!(y_1-j_1)! \tag{1.1}$$

where $(j_1, j_2) \in T; T = \{(j_1, j_2) : j_1 = 0,1,...,y_1; j_2 = 0,1,...,\infty;\}$, $0 < \beta \le 1$, $0 \le p_1, p_2 \le 1$ and $\alpha > 0$. We denote this distribution by BP$(\lambda_1, \lambda_2, \beta, \alpha)$.

1.2 Main Properties of BP $(\lambda_1, \lambda_2, \beta, \alpha)$

Probability generating function (pgf):

$p(t_1, t_2) = \exp((-1/(1+\alpha))(-\lambda_2(1-t_2)(1+\alpha) + \lambda_1\beta(1-t_1)(1+t_2\alpha))$.

Covariance: Covariance $(Y_1, Y_2) = \lambda_1\beta\alpha/(1+\alpha)$, where $\alpha > 0; 0 < \beta \le 1$ and $\lambda_1 < \lambda_2$.

Correlation coefficient: $\rho = \alpha/(1+\alpha))\sqrt{\lambda_1\beta/\lambda_2}$, where $0 < \beta \le 1, \alpha > 0$ and $\lambda_1 < \lambda_2$.

1.3 A re-parameterization of BP$(\lambda_1, \lambda_2, \beta; \alpha)$

A re-parameterization of (1.1) is derived by assuming λ_1 proportional to λ_2 say, $\lambda_1 = a\lambda_2, a \ge 0$ a is a arbitrary constant i.e. $\lambda_2 = \lambda$ and $\lambda_1 = a\lambda$.

The resulting pmf is given by

$$\Pr(Y_1 = y_1, Y_2 = y_2) = \exp(-\lambda)(a\lambda\beta/(1+\alpha))^{y_1}\exp(-a\lambda\beta/(1+\alpha))$$
$$\sum_{(j_1,j_2)\in p}((\alpha)^{j_1}((\lambda\beta - a\alpha\lambda\beta/(1+\alpha))^{j_2}$$
$$(\lambda(1-\beta))^{y_2-j_1-j_2})/j_1!j_2!(y_2-j_1-j_2)!(y_1-j_1)! \quad (1.2)$$

In this article the above bivariate Poisson pmfs in (1.2) and (1.1) have been considered for deriving two new bivariate negative binomial distributions by compounding on the parameter λ and investigate their distributional properties. In section-2 a bivariate negative binomial distribution-I has been defined and its different properties studied. In section-3 another bivariate negative binomial distribution-II has been defined and its different properties investigated.

2. BIVARIATE NEGATIVE BINOMIAL DISTRIBUTION-I

In this section a bivariate negative binomial distribution has been proposed and it properties presented.

Theorem 1 *If* (Y_1, Y_2) *~BP* $(a\lambda, \lambda, \beta, \alpha)$ *in (1.2) and* $\lambda \sim$ *Gamma* (ν, γ) then (Y_1, Y_2) *follows bivariate negative binomial distribution-I having pmf*

$$\Pr(Y_1 = y_1, Y_2 = y_2) = \sum_{(j_1,j_2)\in T}((\alpha)^{j_1}/(j_1!j_2!(y_2-j_1-j_2)!(y_1-j_1)!)$$
$$(a\beta/(1+\alpha))^{y_1}(\beta(1-a\alpha)/(1+\alpha))^{j_2}(1-\beta)^{y_2-j_1-j_2}\{\gamma^{\upsilon}/\Gamma(\upsilon)\}$$
$$\Gamma(y_1+y_2+\nu-j_1)/(1+(a\beta/(1+\alpha))+\gamma)^{y_1+y_2+\nu-j_1} \quad (2.1)$$

where $(j_1, j_2) \in T; T = \{(j_1, j_2) : j_1 = 0,1,\ldots,y_1;\ j_2 = 0,1,\ldots,\infty;\}$, $0 < \beta \le 1$, and $\alpha > 0$.

Proof:

Assume that, $\lambda \sim$ Gamma (ν, γ), with pdf

$g(\lambda) = \{\gamma^{\nu}/\Gamma(\upsilon)\}\lambda^{\upsilon-1}\exp(-\gamma\lambda)$, $\lambda > 0$

The pmf of a bivariate negative binomial distribution is then obtained by compounding (1.2) as

$$\Pr(Y_1 = y_1, Y_2 = y_2) = \int_0^\infty[\exp(-\lambda)(a\lambda\beta/(1+\alpha))^{y_1}\exp(-a\lambda\beta/(1+\alpha))$$
$$\sum_{(j_1,j_2)\in T}((\alpha)^{j_1}((\lambda\beta - a\alpha\lambda\beta/(1+\alpha))^{j_2}(\lambda(1-\beta))^{y_2-j_1-j_2})/$$
$$(j_1!j_2!(y_2-j_1-j_2)!(y_1-j_1)!)g(\lambda)d\lambda]$$
$$= (a\beta/(1+\alpha))^{y_1}\sum_{(j_1,j_2)\in T}((\alpha)^{j_1}/(j_1!j_2!(y_2-j_1-j_2)!(y_1-j_1)!)$$
$$\int_0^\infty[\exp(-\lambda)(\lambda)^{y_1}\exp(-a\lambda\beta/(1+\alpha))((\lambda\beta - a\alpha\lambda\beta/(1+\alpha))^{j_2}$$
$$(\lambda(1-\beta))^{y_2-j_1-j_2})\{\gamma^{\upsilon}/\Gamma(\upsilon)\}\lambda^{\upsilon-1}\exp(-\gamma\lambda)d\lambda]$$
$$= (a\beta/(1+\alpha))^{y_1}\sum_{(j_1,j_2)\in T}((\alpha)^{j_1}/(j_1!j_2!(y_2-j_1-j_2)!(y_1-j_1)!)$$
$$\int_0^\infty[\exp(-\lambda(1+(a\beta/(1+\alpha))+\gamma)(\lambda)^{y_1}((\lambda\beta(1-a\alpha/(1+\alpha)))^{j_2}$$
$$(\lambda(1-\beta))^{y_2-j_1-j_2})\{\gamma^{\upsilon}/\Gamma(\upsilon)\}\lambda^{\upsilon-1}d\lambda]$$
$$= (a\beta/(1+\alpha))^{y_1}\{\gamma^{\upsilon}/\Gamma(\upsilon)\}\sum_{(j_1,j_2)\in T}((\alpha)^{j_1}/(j_1!j_2!(y_2-j_1-j_2)!(y_1-j_1)!)$$
$$(\beta(1-a\alpha)/(1+\alpha))^{j_2}(1-\beta)^{y_2-j_1-j_2}$$

$$\Gamma(y_1+y_2+\nu-j_1)/(1+(a\beta/(1+\alpha))+\gamma)^{y_1+y_2+\nu-j_1}$$

$$=\sum_{(j_1,j_2)\in T}((\alpha)^{j_1}/(j_1!\,j_2!(y_2-j_1-j_2)!(y_1-j_1)!)$$

$$(a\beta/(1+\alpha))^{y_1}(\beta(1-a\alpha)/(1+\alpha))^{j_2}(1-\beta)^{y_2-j_1-j_2}\{\gamma^{\upsilon}/\Gamma(\upsilon)\}$$

$$\Gamma(y_1+y_2+\nu-j_1)/(1+(a\beta/(1+\alpha))+\gamma)^{y_1+y_2+\nu-j_1}$$

This is the required pmf of bivariate negative binomial distribution with parameters α,β,γ and ν //

In particular, when $\alpha=0$, the pmf of bivariate negative binomial distribution in (2.1) reduces to

$$\Pr(Y_1=y_1,Y_2=y_2)=\sum_{(j_1,j_2)\in T}(1/(j_2!(y_2-j_2)!\,y_1!)(a\beta)^{y_1}(\beta)^{j_2}(1-\beta)^{y_2-j_2}\{\gamma^{\upsilon}/\Gamma(\upsilon)\}$$

$$\Gamma(y_1+y_2+\nu)/(1+a\beta+\gamma)^{y_1+y_2+\nu}$$

$$=\sum_{(j_1,j_2)\in T}(y_2!(\beta)^{j_2}(1-\beta)^{y_2-j_2}/(j_2!(y_2-j_2)!))(a\beta)^{y_1}$$

$$\{\gamma^{\upsilon}/\Gamma(\upsilon)\}/y_1!\,y_2!\,\Gamma(y_1+y_2+\nu)/(1+a\beta+\gamma)^{y_1+y_2+\nu}$$

$$=\{\Gamma(\upsilon+y_1+y_2)/\Gamma(y_1+1)\Gamma(\upsilon)\Gamma(y_2+1)\}(a\beta/(1+a\beta+\gamma))^{y_1}$$

$$(\gamma/(1+a\beta+\gamma))^{\nu}(1-(a\beta/(1+a\beta+\gamma))-(\gamma/(1+a\beta+\gamma)))^{y_2} \qquad (2.2)$$

In the rest of this paper, if (Y_1,Y_2) follows bivariate negative binomial distribution with pmf in (2.1) it is symbolically expressed as $(Y_1,Y_2)\sim \text{BNB-I}(\alpha,\beta,\gamma,\nu)$.

2.1 Probability Generating Function (pgf)

Theorem 2 *The joint pgf of* (Y_1,Y_2) *~BNB-I* $(\alpha,\beta,\gamma,\nu)$ *is given by*

$$p(t_1,t_2)=[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon} \qquad (2.3)$$

Proof:

The joint pgf of (Y_1,Y_2) *~BP (* $a\lambda,\lambda,\beta,\alpha$ *)* in (1.2) can seen as

$$p(t_1,t_2)=\exp[-1/(1+\alpha)\{\lambda(1-t_2)(1+\alpha)+a\lambda\beta(1-t_1)(1+t_2\alpha)\}]$$

$$=\exp[-\lambda\{(1-t_2)+a\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}]$$

Now when λ ~ Gamma (ν,γ), then the pgf of compound distribution is given by

$$p(t_1,t_2)=\int_0^{\infty}\exp[-\lambda\{(1-t_2)+a\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}]g(\lambda)d\lambda$$

$$=[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon} \quad // \qquad (2.4)$$

In particular, for $\alpha=0$

$$p(t_1,t_2)=[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)\}]^{-\upsilon} \qquad (2.5)$$

Corollary 1:

From the joint pgf of BNB-I is given in (2.4), the marginal pgfs of Y_1 and Y_2 are respectively obtained as

$$p(t_1)=p(t_1,1)=[1-(1/\beta)\{a(t_1-1)(1+\alpha)/(1+\alpha)\}]^{-\nu}$$

$$=[1-(1/\beta)\{a(t_1-1)\}]^{-\nu}$$

$$p(t_2)=p(1,t_2)=[1-(1/\gamma)\{(t_2-1)\}]^{-\upsilon}$$

Both the marginal pgf are negative binomial as expected.

2.2 Moments and correlations

Theorem 3 *If* $(Y_1,Y_2) \sim \text{BNB-I}(\alpha,\beta,\gamma,\nu)$ then the mean of Y_1 and Y_2 are given by

$$E(Y_1)=a\beta\nu/\gamma$$
$$E(Y_2)=\nu/\gamma$$

Proof:

$$E(Y_1)=(d/dt_1)\{p(t_1,t_{2)}\}\Big|_{t_1=t_2=1}$$
$$=(d/dt_1)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon}]\Big|_{t_1=t_2=1}$$
$$=(-\nu)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1+t_1t_2\alpha-1-t_2\alpha)/(1+\alpha)\}\}^{-\upsilon-1}$$
$$\{(-a\beta-a\beta t_2\alpha)/\gamma(1+\alpha)\}]\Big|_{t_1=t_2=1}$$
$$=(-\nu)\{-a\beta(1+\alpha)/\gamma(1+\alpha)\}$$
$$=a\beta\nu/\gamma \qquad //$$

$$E(Y_2)=(d/dt_2)\{p(t_1,t_{2)}\}\Big|_{t_1=t_2=1}$$
$$=(d/dt_2)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon}]\Big|_{t_1=t_2=1}$$
$$=(-\nu)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1+t_1t_2\alpha-1-t_2\alpha)/(1+\alpha)\}\}^{-\upsilon-1}$$
$$\{-1/\gamma-a\beta/\gamma(t_1\alpha-\alpha)/(1+\alpha)\}]\Big|_{t_1=t_2=1}$$
$$=\ \nu/\gamma \qquad //$$

Theorem 4 *If* $(Y_1,Y_2) \sim \text{BNB-I}(\alpha,\beta,\gamma,\nu)$ then the variances of Y_1 and Y_2 are given by

$$V(Y_1)=a^2\beta^2\nu/\gamma^2+a\beta\nu/\gamma$$
$$V(Y_2)=\nu/\gamma^2+\nu/\gamma$$

Proof:

Now $E(Y_1(Y_1-1)) = (d^2/dt_1{}^2)\{p(t_1,t_{2)}\}\Big|_{t_1=t_2=1}$

$$= (d^2/dt_1{}^2)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon}]\Big|_{t_1=t_2=1}$$
$$= [(-\nu)(-\nu-1)\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon-2}$$
$$\{(-a\beta-a\beta t_2\alpha)/\beta(1+\alpha)\}^2]\Big|_{t_1=t_2=1}$$
$$=(\nu^2+\nu)\{a\beta(1+\alpha)/\gamma(1+\alpha)\}^2$$
$$=a^2\beta^2(\nu^2+\nu)/\gamma^2$$

Therefore, $V(Y_1)=E(Y_1(Y_1-1))+E(Y_1)-(E(Y_1))^2$

$$=\{a^2\beta^2(\nu^2+\nu)/\gamma^2\}+a\beta\nu/\gamma-\{a\beta\nu/\gamma\}^2$$
$$=a^2\beta^2\nu/\gamma^2+a\beta\nu/\gamma \qquad //$$

Again $E(Y_2(Y_2-1))=(d^2/dt_2{}^2)\{p(t_1,t_{2)}\}\Big|_{t_1=t_2=1}$

$$=(d^2/dt_2^{\ 2})[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon}]\Big|_{t_1=t_2=1}$$
$$=[(-\nu)(-\nu-1)\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1+t_1t_2\alpha-1-t_2\alpha)/(1+\alpha)\}\}^{-\upsilon-2}$$
$$\{(-1/\gamma)-a\beta(t_1\alpha-\alpha/\gamma(1+\alpha)\}^2]\Big|_{t_1=t_2=1}$$
$$=[(-\nu)(-\nu-1)\{(-1/\gamma)\}^2]$$
$$=(\nu^2+\nu)(1/\gamma^2)$$

Therefore, $V(Y_2)=E(Y_2(Y_2-1))+E(Y_2)-(E(Y_2))^2$
$$=(\nu^2+\nu)(1/\gamma^2)+\nu/\gamma-\nu^2/\gamma^2$$
$$=\nu/\gamma^2+\nu/\gamma \qquad //$$

Theorem 5 *If* $(Y_1,Y_2)\sim \text{BNB-I}(\alpha,\beta,\gamma,\nu)$ *then the covariance between* Y_1 *and* Y_2 *is given by* $a\beta\nu(\alpha\gamma+\alpha+1)/\gamma^2(1+\alpha)$.

Proof:

$$E(Y_1Y_2)=(d^2/dt_1dt_2)\,p(t_1,t_2)\big|_{t_1=t_2=1}$$
$$=(d^2/dt_1dt_2)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon}]\Big|_{t_1=t_2=1}$$
$$=d/dt_1[(d/dt_2)\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon}]\Big|_{t_1=t_2=1}$$
$$=(-\nu)[\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon-1}\{-a\beta\alpha/\gamma(1+\alpha)\}+$$
$$\{(-1/\gamma)-a\beta t_1\alpha/\gamma(1+\alpha)+a\beta\alpha/\gamma(1+\alpha)\}$$
$$\{(-\nu-1)\{1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}\}^{-\upsilon-2}\}$$
$$\{(-a\beta/\gamma(1+\alpha))-a\beta t_2\alpha/\gamma(1+\alpha)\}]\Big|_{t_1=t_2=1}$$
$$=(-\nu)[\{-a\beta\alpha/\gamma(1+\alpha)\}+\{(-1/\gamma)\{(-\nu-1)(-a\beta/\gamma(1+\alpha))-a\beta\alpha/\gamma(1+\alpha)\}\}$$
$$=(a\beta\alpha\gamma\nu+a\beta\nu^2+a\beta\nu+a\beta\alpha\nu^2+a\beta\alpha\nu)/\gamma^2(1+\alpha)$$

Now $Cov(Y_1,Y_2)=E(Y_1Y_2)-E(Y_1)E(Y_2)$
$$=(a\beta\alpha\nu\gamma+a\beta\nu^2+a\beta\nu+a\beta\alpha\nu^2+a\beta\alpha\nu)/\gamma^2(1+\alpha)-(a\beta\nu^2/\gamma^2)$$
$$=a\beta\nu(\alpha\gamma+\alpha+1)/\gamma^2(1+\alpha) \qquad //$$

Theorem 6 *The correlation coefficient of* (Y_1,Y_2) when (Y_1,Y_2) ~ $\text{BNB-I}(\alpha,\beta,\gamma,\nu)$ *is given by* $\rho=(a\beta\nu(1+\alpha+\gamma\alpha)/(1+\alpha))/(\sqrt{\nu a^2\beta^2+\nu a\beta\gamma})(\sqrt{\nu+\nu\gamma})$.

Proof: Result immediately follows from theorem 5.

Theorem 7 *The joint factorial moments of* $(Y_1, Y_2) \sim \text{BNB - I}(\alpha, \beta, \gamma, \nu)$ *is given by*

$$\mu_{[r,s]} = r!s!(a\beta/\gamma)^r(1/\gamma)^s \sum_{i=0}^{\min(r,s)} (\Gamma(\nu+r+s-i)/(\Gamma(\nu)(r-i)!(s-i)!i!)(\gamma\alpha/(1+\alpha))^i \quad (2.6)$$

where $\mu_{[r,s]} = E[Y_1^{[r]}Y_2^{[s]}]$, and $Y^{[r]} = Y(Y-1)(Y-2)...(Y-r+1)$.

Proof:

It is known that the factorial moment generating function (fmgf) $G(t_1,t_2)$ is related to pgf $p(t_1,t_2)$ as $G(t_1,t_2) = p(t_1+1, t_2+1)$ and joint factorial moments can be derived from fmgf as

$$\mu_{[r,s]} = \partial^{r+s}/(\partial t_1^r \partial t_2^s)\, p(t_1+1, t_2+1)\Big|_{t_1=0,t_2=0}$$

$$= (\partial^{r+s}/\partial t_1^r \partial t_2^s)[1-(1/\gamma)\{(t_2+1-1)+a\beta(t_1+1-1)(1+(t_2+1)\alpha)/(1+\alpha)\}]^{-\upsilon}\Big|_{t_1=0,t_2=0}$$

$$= (\partial^{r+s}/\partial t_1^r \partial t_2^s)[[1-(1/\gamma)\{t_2+(a\beta t_1+a\beta t_1 t_2\alpha+a\beta t_1\alpha)/(1+\alpha)\}]^{-\upsilon}]\Big|_{t_1=0,t_2=0}$$

$\mu_{[r,s]}$ can also be derived as the coefficient of $t_1^r t_2^s / r!s!$ in the power series expansion of the function $G(t_1,t_2)$.

Now $G(t_1,t_2)$ can be rewritten as

$$G(t_1,t_2) = [1-(1/\gamma)\{t_2+(a\beta t_1+a\beta t_1 t_2\alpha+a\beta t_1\alpha)/(1+\alpha)\}]^{-\upsilon}$$

$$= [1-t_1\{(a\beta/\gamma(1+\alpha))+a\beta\alpha/\gamma(1+\alpha)\}-t_2/\gamma-a\beta\alpha t_1 t_2/\gamma(1+\alpha)]^{-\upsilon}$$

$$= [1-t_1(a\beta/\gamma)-t_2/\gamma-a\beta\alpha t_1 t_2/\gamma(1+\alpha)]^{-\upsilon} \quad (2.7)$$

Expanding (2.7) we get

$$G(t_1,t_2) = \sum_{i=0}^{\min(r,s)} (\Gamma(\nu+r-i+s-i+i)/(\Gamma(\nu)(r-i)!(s-i)!i!)(a\beta t_1/\gamma)^{r-i}$$

$$(t_2/\gamma)^{s-i}(a\beta\alpha t_1 t_2/\gamma(1+\alpha))^i$$

$$= \sum_{i=0}^{\min(r,s)} (\Gamma(\nu+r+s-i)/(\Gamma(\nu)(r-i)!(s-i)!i!)(a\beta/\gamma)^r$$

$$t_1^{r-i}(1/\gamma)^s t_2^{s-i}(\gamma\alpha/(1+\alpha))^i t_1^i t_2^i \quad (2.8)$$

Collecting the coefficient of $t_1^r t_2^s / r!s!$ we get

$$\mu_{[r,s]} = r!s!(a\beta/\gamma)^r(1/\gamma)^s \sum_{i=0}^{\min(r,s)} (\Gamma(\nu+r+s-i)/(\Gamma(\nu)(r-i)!(s-i)!i!)(\gamma\alpha/(1+\alpha))^i \quad //$$

In particular

$$\mu_{[1,1]} = (a\beta/\gamma)(1/\gamma)[\Gamma(\nu+1+1)/(\Gamma(\nu))+(\Gamma(\nu+1)/\Gamma(\nu))(\gamma\alpha/(1+\alpha))]$$

$$= (a\beta/\gamma^2)[\nu^2+\nu+(\nu)(\gamma\alpha/(1+\alpha))]$$

$$= [(a\beta\nu^2+a\beta\nu^2\alpha+a\beta\nu+a\beta\nu\alpha+a\beta\gamma\alpha\nu)/\gamma^2(1+\alpha)]$$, which is the $E(Y_1, Y_2)$

$$\mu_{[1,2]} = 2!(a\beta/\gamma)(1/\gamma)^2[\Gamma(\nu+1+2)/(\Gamma(\nu)2!)+(\Gamma(\nu+2)/\Gamma(\nu))(\gamma\alpha/(1+\alpha))]$$

$$= [(\nu+2)(\nu+1)\nu(a\beta/\gamma^3)+2a\beta\alpha\nu(\nu+1)/\gamma^2(1+\alpha))]$$

$$\mu_{[2,1]} = 2!(a\beta/\gamma)^2(1/\gamma)[\Gamma(\nu+1+2)/(\Gamma(\nu)2!)+(\Gamma(\nu+2)/\Gamma(\nu))(\gamma\alpha/(1+\alpha))]$$

$$= [(\nu+2)(\nu+1)\nu(a^2\beta^2/\gamma^3)+2a^2\beta^2\alpha\nu(\nu+1)/\gamma^2(1+\alpha))]$$

$$\mu_{[2,2]} = 2!2!(a\beta/\gamma)^2(1/\gamma)^2[\Gamma(\nu+2+2)/(\Gamma(\nu)2!2!) + (\Gamma(\nu+2+2-1)/\Gamma(\nu))(\gamma\alpha/(1+\alpha))$$
$$+ (\Gamma(\nu+2)/\Gamma(\nu))(\gamma^2\alpha^2/2(1+\alpha)^2)]$$
$$= [(\nu+3)(\nu+2)(\nu+1)\nu(a^2\beta^2/\gamma^4) + 4a^2\beta^2\alpha\nu(\nu+1)(\nu+2)/$$
$$\gamma^3(1+\alpha) + 2\nu(\nu+1)a^2\beta^2\alpha^2/(1+\alpha)^2\gamma^2]$$

2.3 Conditional distributions

Theorem 8 *The conditional pgf of* Y_1 *given* $Y_2 = y_2$ *when* $(Y_1, Y_2) \sim \text{BNB-I}(\alpha,\beta,\gamma,\nu)$ *is given by*

$$p_{Y_1}(t/y_2) = [\{1-(1/\gamma)\{(-1)+a\beta(t-1)/(1+\alpha)\}\}^{-\upsilon-y_2}\{(-1/\gamma)-(a\beta t\alpha/\gamma(1+\alpha))$$
$$+ a\beta\alpha/\gamma(1+\alpha)\}^{y_2}]/[1+(1/\gamma)]^{-\upsilon-y_2}\{(-1/\gamma)\}^{y_2} \qquad (2.9)$$

Proof:

The pgf of the conditional distribution of Y_1 given $Y_2 = y_2$ has been derived using the formula

$p_{Y_1}(t/y_2) = p^{(0,y_2)}(t,0)/p^{(0,y_2)}(1,0)$ (Kocherlakota and Kocherlakota, 1992).

where, $p^{(y_1,y_2)}(u,v) = (\partial^{y_1+y_2}/\partial t_1^{y_1} t_2^{y_2})\, p(t_1,t_2)\Big|_{t_1=u, t_2=v}$, $p(t_1,t_2)$ is the pgf of BNB-I(α,β,γ,ν) given in (2.3).

Now,

$$\frac{\partial}{\partial t_2} p(t_1,t_2) = \frac{\partial}{\partial t_2}[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon}$$
$$= (-\nu)[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon-1} \qquad (2.10)$$

Proceeding similarly differentiating $p(t_1,t_2)$ y_2 times w.r.t t_2 we get

$$\frac{\partial^{y_2}}{\partial t_2^{y_2}} p(t_1,t_2) = (-\nu)(-\nu-1)\ldots(-\nu-y_2+1)[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon-y_2}$$
$$\{(-1/\gamma)-(a\beta t_1\alpha/\gamma(1+\alpha))+a\beta\alpha/\gamma(1+\alpha)\}^{y_2}$$

Now $p^{(0,y_2)}(t,0) = \dfrac{\partial^{y_2}}{\partial t_2^{y_2}} p(t_1,t_2)\Big|_{t_1=t, t_2=0}$

$$= (-\nu)(-\nu-1)\ldots(-\nu-y_2+1)[1-(1/\gamma)\{(-1)+a\beta(t-1)/(1+\alpha)\}]^{-\upsilon-y_2}$$
$$\{(-1/\gamma)-(a\beta t\alpha/\gamma(1+\alpha))+a\beta\alpha/\gamma(1+\alpha)\}^{y_2}$$

Hence $p^{(0,y_2)}(1,0) = (-\nu)(-\nu-1)\ldots(-\nu-y_2+1)[1-(1/\gamma)\{(-1)\}]^{-\upsilon-y_2}\{(-1/\gamma)\}^{y_2}$

Thus the pgf of conditional distribution of Y_1 given $Y_2 = y_2$ is given by

$$p_{y_1}(t/y_2) = [\{1-(1/\gamma)\{(-1)+a\beta(t-1)/(1+\alpha)\}\}^{-\upsilon-y_2}$$
$$\{(-1/\gamma)-(a\beta t\alpha/\gamma(1+\alpha))+a\beta\alpha/\gamma(1+\alpha)\}^{y_2}]/[1+(1/\gamma)]^{-\upsilon-y_2}\{(-1/\gamma)\}^{y_2} \quad //$$

Theorem 9 *The conditional pgf of* Y_2 *given* $Y_1 = y_1$ *when* $(Y_1, Y_2) \sim \text{BNB-I}(\alpha,\beta,\gamma,\nu)$ *is given by*

$$p_{Y_2}(t/y_1) = \{[1-(1/\gamma)\{(t-1)+a\beta(-1)(1+t\alpha)/(1+\alpha)\}\}^{-\upsilon-y_1}$$
$$\{-a\beta(1+t\alpha)/\gamma(1+\alpha)\}^{y_1}]/[(-a\beta/\gamma)^{y_1}+(a\beta/\gamma))^{-\upsilon-y_1}] \qquad (2.11)$$

Proof:

The pgf of the conditional distribution of Y_2 given $Y_1 = y_1$ has been derived using the formula
$p_{Y_2}(t/y_1) = p^{(y_1,0)}(0,t)/p^{(y_1,0)}(0,1)$ (Kocherlakota and Kocherlakota, 1992).

where, $p^{(y_1,y_2)}(u,v) = (\partial^{y_1+y_2} / \partial t_1^{y_1} t_2^{y_2})\, p(t_1,t_2)\Big|_{t_1=u,t_2=v}$, $p(t_1,t_2)$ is the pgf of *BNB-I*(α,β,γ,ν) given in (2.3).

Now,

$$\frac{\partial}{\partial t_1} p(t_1,t_2) = \frac{\partial}{\partial t_1}[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon}$$

$$=(-\nu)[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon-1}$$
$$\{-a\beta(1+t_2\alpha)/\gamma(1+\alpha)\}$$

Differentiating $p(t_1,t_2)$ y_1 times w.r.t t_1 we get

$$\frac{\partial^{y_1}}{\partial t_1^{y_1}} p(t_1,t_2) = (-\nu)(-\nu-1)...(-\nu-y_1+1)[1-(1/\gamma)\{(t_2-1)+a\beta(t_1-1)(1+t_2\alpha)/(1+\alpha)\}]^{-\upsilon-y_1}$$
$$\{-a\beta(1+t_2\alpha)/\gamma(1+\alpha)\}^{y_1}$$

Now $p^{(y_1,0)}(0,t) = \frac{\partial^{y_1}}{\partial t_1^{y_1}} p(t_1,t_2)\Big|_{t_1=0,t_2=t}$

$$=(-\nu)(-\nu-1)...(-\nu-y_1+1)[1-(1/\gamma)\{(t-1)+a\beta(-1)(1+t\alpha)/(1+\alpha)\}]^{-\upsilon-y_1}$$
$$\{-a\beta(1+t\alpha)/\beta(1+\alpha)\}^{y_1}$$
$$=(-\nu)(-\nu-1)...(-\nu-y_1+1)[1-(1/\gamma)\{(t-1)-a\beta(1+t\alpha)/(1+\alpha)\}]^{-\upsilon-y_1}$$
$$\{-a\beta(1+t\alpha)/\gamma(1+\alpha)\}^{y_1}$$

Therefore $p^{(y_1,0)}(0,1) = \frac{\partial^{y_1}}{\partial t_1^{y_1}} p(t_1,t_2)\Big|_{t_1=0,t_2=1}$

$$=(-\nu)(-\nu-1)...(-\nu-y_1+1)[1-(1/\gamma)\{a\beta(-1)(1+\alpha)/(1+\alpha)\}]^{-\upsilon-y_1}\{-a\beta/\gamma\}^{y_1}$$

Thus the pgf of conditional distribution of Y_1 given $Y_2 = y_2$ is given by

$$p_{y_2}(t/y_1) = \{[1-(1/\gamma)\{(t-1)+a\beta(-1)(1+t\alpha)/(1+\alpha)\}\}^{-\upsilon-y_1}$$
$$\{-a\beta(1+t\alpha)/\gamma(1+\alpha)\}^{y_1}]/[(-a\beta/\gamma)^{y_1}+(a\beta/\gamma))^{-\upsilon-y_1}] \qquad //$$

3. BIVARIATE NEGATIVE BINOMIAL DISTRIBUTION-II

In this section another bivariate negative binomial distribution has been proposed and its properties presented.

Theorem 10 *If (Y_1,Y_2) ~BP ($\lambda_1,\lambda_2,\beta,\alpha$) in (1.1) and λ_1~ Gamma (ν,γ_1) and λ_2~ Gamma (ν,γ_2) then the resulting compounding distribution is the bivariate negative binomial distribution-II having pmf*

$$\Pr(Y_1 = y_1, Y_2 = y_2) = \{\gamma_1{}^{\upsilon}/\Gamma(\upsilon)\}\{\gamma_2{}^{\upsilon}/\Gamma(\upsilon)\}\sum_{i=0}^{j_2}{}^{j_2}c_i(\Gamma(y_1+j_2+\upsilon-i)/$$
$$((\beta)^i((\beta/(1+\alpha))+\gamma_1)^{y_1+j_2+\upsilon-i}\sum_{(j_1,j_2)\in p}((\alpha)^{j_1}(\alpha\beta/(1+\alpha))^{j_2-i}$$
$$(\beta/(1+\alpha)^{y_1}(1-\beta)^{y_2-j_1-j_2}/(j_2!\,j_2!\,y_1!(y_2-j_1-j_2)!(y_1-j_1)!)$$
$$\Gamma(y_2+i+\nu-j_1-j_2)/(1+\gamma_2)^{y_2+i+\nu-j_1-j_2} \qquad (3.1)$$

Proof:

Assume that, $\lambda_1 \sim$ Gamma (ν, γ_1) and $\lambda_2 \sim$ Gamma (ν, γ_2) with respective pdf

$g(\lambda_1) = \{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}\lambda_1^{\upsilon-1} \exp(-\gamma_1\lambda_1)$, $\lambda_1 > 0$ and

$g(\lambda_2) = \{\gamma_2^{\upsilon} / \Gamma(\upsilon)\}\lambda_2^{\upsilon-1} \exp(-\gamma_2\lambda_2)$, $\lambda_2 > 0$

Now the pmf of Bivariate Negative binomial distribution is obtained by compounding from BP $(\lambda_1, \lambda_2, \beta; \alpha)$ in (1.1) as

$$\Pr(Y_1 = y_1, Y_2 = y_2) = \int_0^{\infty}\int_0^{\infty} \exp(-\lambda_2)(\lambda_1\beta/(1+\alpha))^{y_1} \exp(-\lambda_1\beta/(1+\alpha))$$
$$\sum_{(j_1,j_2)\in T} ((\alpha)^{j_1} ((\lambda_2\beta - \alpha\lambda_1\beta/(1+\alpha))^{j_2}$$
$$(\lambda_2(1-\beta))^{y_2-j_1-j_2})/(j_1!\, j_2!(y_2 - j_1 - j_2)!$$
$$(y_1 - j_1)!)g(\lambda_1)g(\lambda_2)d\lambda_1 d\lambda_2$$
$$= \int_0^{\infty} \sum_{(j_1,j_2)\in T} ((\alpha)^{j_1} /(j_2!\, j_2!\, y_1!(y_2 - j_1 - j_2)!(y_1 - j_1)!)$$
$$\sum_{i=0}^{j_2} {}^{j_2}c_i (\lambda_2\beta)^i \exp(-\lambda_2)(\lambda_2(1-\beta))^{y_2-j_1-j_2} g(\lambda_2)d\lambda_2$$
$$\int_0^{\infty} (\lambda_1\beta/(1+\alpha))^{y_1} (\alpha\lambda_1\beta/(1+\alpha))^{j_2-i} \exp\{(-\lambda_1\beta)/(1+\alpha)\}$$
$$\{\gamma_1^{\upsilon}/\Gamma(\upsilon)\}\lambda_1^{\upsilon-1}\exp(-\gamma_1\lambda_1)d\lambda_1$$
$$= \{\gamma_1^{\upsilon}/\Gamma(\upsilon)\}\int_0^{\infty} \sum_{(j_1,j_2)\in T} ((\alpha)^{j_1} (\alpha\beta/(1+\alpha))^{j_2-i} (\beta/(1+\alpha)^{y_1} /$$
$$(j_2!\, j_2!\, y_1!(y_2 - j_1 - j_2)!(y_1 - j_1)!)$$
$$\sum_{i=0}^{j_2} {}^{j_2}c_i (\lambda_2\beta)^i \exp(-\lambda_2)(\lambda_2(1-\beta))^{y_2-j_1-j_2}$$
$$\int_0^{\infty} \exp\{(-\lambda_1)((\beta/(1+\alpha)) + \gamma_1\}\lambda_1^{y_1+j_2+\upsilon-i-1} d\lambda_1 \; g(\lambda_2)d\lambda_2$$
$$= \{\gamma_1^{\upsilon}/\Gamma(\upsilon)\}\int_0^{\infty} \sum_{(j_1,j_2)\in T} ((\alpha)^{j_1} (\alpha\beta/(1+\alpha))^{j_2-i} (\beta/(1+\alpha)^{y_1} /$$
$$(j_2!\, j_2!\, y_1!(y_2 - j_1 - j_2)!(y_1 - j_1)!)$$
$$\sum_{i=0}^{j_2} {}^{j_2}c_i (\lambda_2\beta)^i \exp(-\lambda_2)(\lambda_2(1-\beta))^{y_2-j_1-j_2} g(\lambda_2)d\lambda_2$$
$$(\Gamma(y_1 + j_2 + \upsilon - i)/(((\beta/(1+\alpha)) + \gamma_1)^{y_1+j_2+\upsilon-i}$$
$$= \{\gamma_1^{\upsilon}/\Gamma(\upsilon)\} \sum_{(j_1,j_2)\in T} ((\alpha)^{j_1} (\alpha\beta/(1+\alpha))^{j_2-i} (\beta/(1+\alpha)^{y_1} /$$
$$(j_2!\, j_2!\, y_1!(y_2 - j_1 - j_2)!(y_1 - j_1)!)$$
$$\sum_{i=0}^{j_2} {}^{j_2}c_i (\Gamma(y_1 + j_2 + \upsilon - i)/(((\beta/(1+\alpha)) + \gamma_1)^{y_1+j_2+\upsilon-i}$$

$$(\beta)^i (1-\beta)^{y_2-j_1-j_2} \{\gamma_2^{\upsilon} / \Gamma(\upsilon)\}$$

$$\int_0^\infty \exp(-\lambda_2(1+\gamma_2))\lambda_2^{y_2+i+\nu-j_1-j_2-1} d\lambda_2$$

$$=\{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}\{\gamma_2^{\upsilon} / \Gamma(\upsilon)\}\sum_{i=0}^{j_2} {}^{j_2}c_i (\Gamma(y_1 + j_2 + \upsilon - i) /$$

$$(((\beta/(1+\alpha)) + \gamma_1)^{y_1+j_2+\upsilon-i} (\beta)^i \sum_{(j_1,j_2)\in T} ((\alpha)^{j_1} (\alpha\beta/(1+\alpha))^{j_2-i}$$

$$(\beta/(1+\alpha)^{y_1} (1-\beta)^{y_2-j_1-j_2} / (j_2!\, j_2!\, y_1!(y_2 - j_1 - j_2)!(y_1 - j_1)!)$$

$$\Gamma(y_2 + i + \nu - j_1 - j_2)/(1+\gamma_2)^{y_2+i+\nu-j_1-j_2} \qquad //$$

In the rest of this paper, if (Y_1, Y_2) follows bivariate negative binomial distribution with pmf in (3.1) it is symbolically expressed as $(Y_1, Y_2) \sim \text{BNB-II}(\alpha, \beta, \gamma_1, \gamma_2)$.

3.1 Probability Generating Function (pgf)

Theorem 11 *The pgf of* $\text{BNB-II}(\alpha, \beta, \gamma_1, \gamma_2)$ *is given by*

$$p(t_1, t_2) = [\{1 - \{\beta(t_1 - 1)(1 + t_2\alpha)\}\} / \gamma_1(1+\alpha)\}\{1 - (t_2 - 1)/\gamma_2\}]^{-\nu} \qquad (3.2)$$

Proof:

The joint pgf of (Y_1, Y_2)~$BP(\lambda_1, \lambda_2, \beta, \alpha)$ is given by

$$p(t_1, t_2) = \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha) + \lambda_1\beta(1-t_1)(1+t_2\alpha)\}]$$

Now since λ_1 ~ Gamma (ν, γ_1) and λ_2 ~ Gamma (ν, γ_2), the pgf of the compound distribution is obtained as

$$p(t_1, t_2) = \int_0^\infty\int_0^\infty \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha) + \lambda\beta_1(1-t_1)(1+t_2\alpha)\}] g(\lambda_1) g(\lambda_2) d\lambda_1 d\lambda_2$$

$$= \int_0^\infty\int_0^\infty \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha) + \lambda_1\beta(1-t_1)(1+t_2\alpha)\}]$$

$$\{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}\lambda_1^{\upsilon-1} \exp(-\gamma_1\lambda_1) d\lambda_1 g(\lambda_2) d\lambda_2$$

$$= \int_0^\infty \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha)\}] g(\lambda_2)$$

$$[\int_0^\infty \exp[-1/(1+\alpha)\{\lambda_1\beta(1-t_1)(1+t_2\alpha)\}]$$

$$\{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}\lambda_1^{\upsilon-1} \exp(-\gamma_1\lambda_1) d\lambda_1] d\lambda_2$$

$$= \int_0^\infty \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha)\}] g(\lambda_2) \{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}$$

$$[\int_0^\infty \exp[\{(-\lambda_1)\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\} + \gamma_1]\lambda_1^{\upsilon-1} d\lambda_1] d\lambda_2$$

$$= \{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}\int_0^\infty \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha)\}] g(\lambda_2)$$

$$[\Gamma(\nu)/[\{\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}] + \gamma_1]^{\nu} d\lambda_2$$

$$= \{\gamma_1^{\upsilon} / \Gamma(\upsilon)\}[\Gamma(\nu)/[\{\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}] + \gamma_1]^{\nu}$$

$$\int_0^\infty \exp[-1/(1+\alpha)\{\lambda_2(1-t_2)(1+\alpha)\}]\{\gamma_2{}^{\upsilon}/\Gamma(\upsilon)\}\lambda_2{}^{\upsilon-1}\exp(-\gamma_2\lambda_2)d\lambda_2$$

$$=\{\gamma_1{}^{\upsilon}/\Gamma(\upsilon)\}[\Gamma(\nu)/[\{\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}]+\gamma_1]^{\nu}$$

$$\{\gamma_2{}^{\upsilon}/\Gamma(\upsilon)\}[\Gamma(\nu)/[\{(1-t_2)(1+\alpha)/(1+\alpha)\}]+\gamma_2]^{\nu}$$

$$=\gamma_1{}^{\upsilon}\gamma_2{}^{\upsilon}[\{\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}+\gamma_1]^{-\nu}[\{(1-t_2)(1+\alpha)/(1+\alpha)\}+\gamma_2]^{-\nu}$$

$$=(\gamma_1\gamma_2)^{\upsilon}[\{\beta(1-t_1)(1+t_2\alpha)/(1+\alpha)\}+\gamma_1]^{-\nu}[1-t_2+\gamma_2]^{-\nu}$$

$$=[\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}]^{-\nu} \qquad //$$

Corollary 2

From the joint pgf of BNB-II given in (3.2) the marginal pgfs of Y_1 and Y_2 are respectively obtained as

$$p(t_1)=p(t_1,1)=[\{1-\{\beta(t_1-1)(1+\alpha)\}\}/\gamma_1(1+\alpha)\}]^{-\nu}$$

$$=[1-\beta(t_1-1)/\gamma_1]^{-\nu}$$

$$p(t_2)=p(1,t_2)=[1-(t_2-1)/\gamma_2]^{-\nu}$$

Thus Y_1 and Y_2 are both negative binomial distribution as expected.

3.2 Moments and correlations

Theorem 12 *If* $(Y_1,Y_2)\sim \text{BNB-II}(\alpha,\beta,\gamma_1,\gamma_2)$ then the mean of Y_1 and Y_2 are given by

$$E(Y_1)=\beta\nu/\gamma_1$$

$$E(Y_2)=\nu/\gamma_2$$

Proof:

$$E(Y_1)=(d/dt_1)\{p(t_1,t_{2)}\}\Big|_{t_1=t_2=1}$$

$$=(d/dt_1)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu}]\Big|_{t_1=t_2=1}$$

$$=(-\nu)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-1}$$

$$\{(-\beta/\gamma_1(1+\alpha))-\beta t_2\alpha/\gamma_1(1+\alpha)\}]\Big|_{t_1=t_2=1}$$

$$=(-\nu)[\{(-\beta/\gamma_1(1+\alpha))-\alpha/\gamma_1(1+\alpha)\}]$$

$$=\nu\beta/\gamma_1 \qquad //$$

$$E(Y_2)=(d/dt_2)\{p(t_1,t_{2)}\}\Big|_{t_1=t_2=1}$$

$$=(d/dt_2)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu}]\Big|_{t_1=t_2=1}$$

$$=(-\nu)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-1}$$

$$\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}(-1/\gamma_2)+$$

$$\{1-(t_2-1)/\gamma_2\}(-\beta t_1\alpha/\gamma_1(1+\alpha))+\alpha\beta/\gamma_1(1+\alpha)]\Big|_{t_1=t_2=1}$$

$$=(-\nu)[(-1/\gamma_2)]$$

$$=\nu/\gamma_2 \qquad //$$

Theorem 13 *If* $(Y_1,Y_2)\sim \text{BNB-II}(\alpha,\beta,\gamma_1,\gamma_2)$ then the variances of Y_1 and Y_2 are given by

$$V(Y_1)=\beta^2\nu/\gamma_1^2+\beta\nu/\gamma_1$$

$V(Y_2) = \nu / \gamma_2^2 + \nu / \gamma_2$

Proof:

Now $E(Y_1(Y_1-1)) = (d^2 / dt_1^{\ 2})\{p(t_1, t_{2)}\}\Big|_{t_1=t_2=1}$

$$= (d^2 / dt_1^{\ 2})[\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}]^{-\nu}\Big|_{t_1=t_2=1}$$

$$= (d / dt_1)[(-\nu)\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-1}$$
$$\{(-\beta/\gamma_1(1+\alpha)) - \beta t_2\alpha/\gamma_1(1+\alpha)\}]\Big|_{t_1=t_2=1}$$

$$= (-\nu)\{(-\beta/\gamma_1(1+\alpha)) - \beta t_2\alpha/\gamma_1(1+\alpha)\}[(-\nu-1)$$
$$\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-2}\{(-\beta/\gamma_1(1+\alpha))$$
$$-\beta t_2\alpha/\gamma_1(1+\alpha)\}]\Big|_{t_1=t_2=1}$$

$$= (\nu^2+\nu)\{-\beta/\gamma_1\}^2$$

$$= (\nu^2+\nu)\{\beta/\gamma_1^{\ 2}\}$$

Therefore, $V(Y_1) = E(Y_1(Y_1-1)) + E(Y_1) - (E(Y_1))^2$

$$= (\nu^2+\nu)\{\beta/\gamma_1^{\ 2}\} + \nu\beta/\gamma_1 - (\nu\beta/\gamma_1)^2$$

$$= (\nu\beta/\gamma_1^{\ 2}) + \nu\beta/\gamma_1 \qquad //$$

Again, $E(Y_2(Y_2-1)) = (d^2 / dt_2^{\ 2})\{p(t_1, t_{2)}\}\Big|_{t_1=t_2=1}$

$$= (d^2 / dt_2^{\ 2})[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}$$
$$= \{1-(t_2-1)/\gamma_2\}\}^{-\nu}]\Big|_{t_1=t_2=1}$$

$$= (d / dt_2)[(-\nu)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-1}$$
$$\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}(-1/\gamma_2) +$$
$$\{1-(t_2-1)/\gamma_2\}(-t_1\alpha\beta/\gamma_1(1+\alpha)) + \alpha\beta/\gamma_1(1+\alpha)]]\Big|_{t_1=t_2=1}$$

$$= (-\nu)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-1}$$
$$\{(-1/\gamma_2)((-\beta t_1\alpha/\gamma_1(1+\alpha)) + \alpha\beta/\gamma_1(1+\alpha) + (-1/\gamma_2)$$
$$(-t_1\alpha\beta/\gamma_1(1+\alpha)) + \alpha\beta/\gamma_1(1+\alpha)\} + \{\{1-\{\beta(t_1-1)$$
$$(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}(-1/\gamma_2) + \{1-(t_2-1)/\gamma_2\}\{(-\beta t_1\alpha/\gamma_1(1+\alpha))$$
$$+\alpha\beta/\gamma_1(1+\alpha)\}\}(-\nu-1)\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}$$
$$\{1-(t_2-1)/\gamma_2)\}\}^{-\nu-2}\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}(-1/\gamma_2) +$$
$$\{1-(t_2-1)/\gamma_2\}(-\beta t_1\alpha/\gamma_1(1+\alpha)) + \alpha\beta/\gamma_1(1+\alpha)]\Big|_{t_1=t_2=1}$$

$$= (-\nu)[(-1/\gamma_2)(-\nu-1)(-1/\gamma_2)]$$

$$= [(-1/\gamma_2)^2(\nu^2+\nu)]$$

$$= (\nu^2+\nu)/\gamma_2^2$$

Therefore, $V(Y_2) = E(Y_2(Y_2-1)) + E(Y_2) - (E(Y_2))^2$

$$=((\nu^2+\nu)/\gamma_2^2)+(\nu/\gamma_2)-(\nu/\gamma_2)^2$$
$$=(\nu/\gamma_2{}^2)+\nu/\gamma_2 \qquad //$$

Theorem 14 *If* $(Y_1,Y_2)\sim \text{BNB-II}(\alpha,\beta,\gamma_1,\gamma_2)$ *then the covariance between* Y_1 *and* Y_2 *is given by* $\{\nu\alpha\beta/\gamma_1(1+\alpha)\}+\{\nu\beta/\gamma_1\gamma_2\}$.

Proof:

$$E(Y_1Y_2)=(d^2/dt_1dt_2)\,p(t_1,t_2)\big|_{t_1=t_2=1}$$
$$=(d^2/dt_1dt_2)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu}]\big|_{t_1=t_2=1}$$
$$=(d/dt_2)[(d/dt_1)\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}$$
$$\{1-(t_2-1)/\gamma_2\}\}^{-\nu}]\big|_{t_1=t_2=1}$$
$$=(-\nu)[\{\{1-\{(t_1-1)(1+t_2\alpha)\}\}/\beta_1(1+\alpha)\}\{1-(t_2-1)/\beta_2\}\}^{-\nu-1}$$
$$\{-\alpha/\beta_1(1+\alpha)\}+\{(-\beta/\gamma_1(1+\alpha))-(\beta t_2\alpha/\gamma_1(1+\alpha)\}(-\nu-1)$$
$$\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-2}$$
$$\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}(-1/\gamma_2)+$$
$$\{1-(t_2-1)/\gamma_2\}(-\beta t_1\alpha/\gamma_1(1+\alpha))+\alpha\beta/\gamma_1(1+\alpha)]\big|_{t_1=t_2=1}$$
$$=(-\nu)[\{-\alpha\beta/\gamma_1(1+\alpha)\}+(-\nu-1)\{(-\beta/\gamma_1(1+\alpha))-(\alpha\beta/\gamma_1(1+\alpha))\}$$
$$\{(-1/\gamma_2)\}]$$
$$=(-\nu)[\{-\alpha\beta/\gamma_1(1+\alpha)\}+(-\nu-1)\{(-\beta/\gamma_1(1+\alpha))(1+\alpha)\}(-1/\gamma_2)]$$
$$=\{\nu\alpha\beta/\gamma_1(1+\alpha)\}+\{(\nu^2+\nu)/\gamma_1\gamma_2\}$$

Now $Cov(Y_1,Y_2)=E(Y_1Y_2)-E(Y_1)E(Y_2)$

$$=\{\nu\alpha\beta/\gamma_1(1+\alpha)\}+\{(\nu^2+\nu)/\gamma_1\gamma_2\}-(\nu\beta/\gamma_1)(\nu/\gamma_2)$$
$$=\{\nu\alpha\beta/\gamma_1(1+\alpha)\}+\{(\nu^2+\nu)/\gamma_1\gamma_2\}-(\nu^2/\gamma_1\gamma_2)$$
$$=\{\nu\alpha\beta/\gamma_1(1+\alpha)\}+\{\nu\beta/\gamma_1\gamma_2\} \qquad //$$

Theorem 15 *The correlation coefficient of* (Y_1,Y_2) for (Y_1,Y_2) ~ $\text{BNB-II}(\alpha,\beta,\gamma_1,\gamma_2)$ *is given by* $\rho=\{\nu\alpha\beta/\gamma_1(1+\alpha)\}+\{\nu\beta/\gamma_1\gamma_2\}/(\sqrt{(\nu\beta^2/\gamma_1^2)+(\nu\beta/\gamma_1)})(\sqrt{(\nu/\gamma_2^2)+(\nu/\gamma_2)})$

Proof: Result immediately follows from theorem 14.

3.3 Conditional distribution

Theorem 16 *The conditional pgf of* Y_2 *given* $Y_1=y_1$ *when* $(Y_1,Y_2)\sim$ $\text{BNB-II}(\alpha,\beta,\gamma_1,\gamma_2)$ *is given by*

$$p_{y_2}(t/y_1)=[\{\{1+\{\beta(1+t\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t-1)/\gamma_2\}\}]^{-\upsilon-y_1}$$
$$\{(-\beta/\gamma_1(1+\alpha))(1+t\alpha)\}^{y_1}/[(1+(\beta/\gamma_1))^{-\upsilon-y_1}(-\beta/\gamma_1)^{y_1}] \qquad (3.3)$$

Proof:

The pgf of the conditional distribution of Y_2 given $Y_1=y_1$ has been derived using the formula

$p_{Y_2}(t/y_1)=p^{(y_1,0)}(0,t)/p^{(y_1,0)}(0,1)$ (Kocherlakota and Kocherlakota, 1992).

where, $p^{(y_1,y_2)}(u,v)=(\partial^{y_1+y_2}/\partial t_1^{y_1}t_2^{y_2})\,p(t_1,t_2)\big|_{t_1=u,t_2=v}$, $p(t_1,t_2)$ is the pgf of BNB-II($\alpha,\beta,\gamma_1,\gamma_2$) given in (3.2).

Now,

$$\frac{\partial}{\partial t_1} p(t_1,t_2) = \frac{\partial}{\partial t_1}[\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}]^{-\nu}$$

$$= (-\nu)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t_2-1)/\gamma_2\}\}^{-\nu-1}$$

$$\{(-\beta/\gamma_1(1+\alpha))-\beta t_2\alpha/\gamma_1(1+\alpha)\}] \quad (3.4)$$

Proceeding similarly differentiating $p(t_1,t_2)$ y_1 times w.r.t t_1 we get

$$\frac{\partial^{y_1}}{\partial t_1^{y_1}} p(t_1,t_2) = (-\nu)(-\nu-1)...(-\nu-y_1+1)[\{\{1-\{\beta(t_1-1)(1+t_2\alpha)\}\}/\gamma_1(1+\alpha)\}$$

$$\{1-(t_2-1)/\gamma_2\}\}]^{-\upsilon-y_1}\{(-\beta/\gamma_1(1+\alpha))-\beta t_2\alpha/\gamma_1(1+\alpha)\}]^{y_1}$$

Now $p^{(y_1,0)}(0,t) = \left.\frac{\partial^{y_1}}{\partial t_1^{y_1}} p(t_1,t_2)\right|_{t_1=0,t_2=t}$

$$= (-\nu)(-\nu-1)...(-\nu-y_1+1)[\{\{1+\{\beta(1+t\alpha)\}\}/\gamma_1(1+\alpha)\}$$

$$\{1-(t-1)/\gamma_2\}\}]^{-\upsilon-y_1}\{(-\beta/\gamma_1(1+\alpha))-\beta t\alpha/\gamma_1(1+\alpha)\}^{y_1}$$

$$= (-\nu)(-\nu-1)...(-\nu-y_1+1)[\{\{1+\{\beta(1+t\alpha)\}\}/\gamma_1(1+\alpha)\}$$

$$\{1-(t-1)/\gamma_2\}\}]^{-\upsilon-y_1}\{(-\beta/\gamma_1(1+\alpha))(1+t\alpha)\}^{y_1}$$

$$p^{(y_1,0)}(0,1) = \left.\frac{\partial^{y_1}}{\partial t_1^{y_1}} p(t_1,t_2)\right|_{t_1=0,t_2=1}$$

$$= (-\nu)(-\nu-1)...(-\nu-y_1+1)[\{\{1+\{\beta(1+\alpha)\}\}/\gamma_1(1+\alpha)\}\}]^{-\upsilon-y_1}$$

$$\{(-\beta/\gamma_1(1+\alpha))-\beta\alpha/\gamma_1(1+\alpha)\}^{y_1}$$

$$= (-\nu)(-\nu-1)...(-\nu-y_1+1)[1+(\beta/\gamma_1)]^{-\upsilon-y_1}(-\beta/\gamma_1)^{y_1}$$

Thus the pgf of conditional distribution of Y_1 given $Y_2 = y_2$ is given by

$$p_{y_2}(t/y_1) = [\{\{1+\{\beta(1+t\alpha)\}\}/\gamma_1(1+\alpha)\}\{1-(t-1)/\gamma_2\}\}]^{-\upsilon-y_1}$$

$$\{(-\beta/\gamma_1(1+\alpha))(1+t\alpha)\}^{y_1}/[(1+(\beta/\gamma_1))^{-\upsilon-y_1}(-\beta/\gamma_1)^{y_1}] \quad //$$

4. CONCLUSION

Two new bivariate negative binomial distributions have been proposed as compound distributions of the recently investigated bivariate Poisson distribution derived by Kotoky and Chakraborty (2013). Some of its important distributional properties have been investigated. Further work is in progress to develop numerical routine for probability calculation, parameter estimation and sample generation.

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TARDINESS OF JOBS AND SATISFACTION LEVEL OF DEMAND MAKER IN m-STAGE SCHEDULING WITH FUZZY DUE TIME

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ABSTRACT :
This paper investigates a general flow shop scheduling problem under fuzzy environment. The objective of the paper is twofold, i.e. to minimize the tardiness of the jobs on one side while to find the satisfaction level of demand maker on other side. The membership function of fuzzy due date assigns to each job indicates the satisfaction level of demand maker i.e. how much percent the customer is satisfied about the service within the time frame. The study has been made through heuristic application and the results are presented graphically which is helpful for the goodwill of the company when there is huge demand in the market.
Keywords: ***Fuzzy processing time, trapezoidal, tardiness, <AHR>.***

INTRODUCTION:

The study of scheduling on machines is an open field for researchers, planners and managers. A good scheduling strategy may help the companies to respond the market demand quickly and to run the industries more efficiently and competitively in the market. In conventional approach of flow shop scheduling, the processing time of jobs has been taken deterministic while in real world situation, the processing time of jobs could be imprecise because of either incomplete knowledge of domain or uncertain environment. In this paper, we assume that the sort of uncertainty that comes into play is better represented by the notion of fuzziness instead of randomness and probability. In complex flow shop scheduling, the concepts of fuzziness are very effective e.g. fuzzy due date can be defined to express the satisfaction degree of decision maker for completion time of a job. Fuzzy sets are often used to describe the processing time when the knowledge about the processing time is incomplete or it will change within an interval with the change of environment. In many situations, the decision maker could only estimate the interval in which processing time lies or give the processing time with a certain degree of confidence level. The concept of fuzzy processing time, fuzzy due date, fuzzy precedence in the field of scheduling, have been studied by various researchers. Chong et. al.(1955), Conway(1965), Mohd. Ikram(1986) discussed about earliness and lateness of jobs in flow shop scheduling. Mc. Cahon & Lee (1992), Ishii (1995), Schibuchi et.al.(1996) have studied fuzzified scheduling problems by using the concept of fuzzy due date. T.P.Singh, Sunita(2008,2010) made an attempt to find the satisfaction level of demand maker using triangular fuzzy rule for two and three stage scheduling problems. Recently, Meenu,T.P. Singh & Deepak Gupta(2014, 2015) studied the bi-objective criteria in two and three stage flow shop scheduling by using the concept of fuzzy due date. This paper is the extension of our earlier work for general scheduling problem. The fuzzy environment has been considered trapezoidal in nature and graded mean integration representation has been applied. Membership function, assigned to each job, represents the satisfaction grade of human user for the completion time of the job.

The Organization of the paper is as follows:
In section 2 we provide a brief introduction about fuzzy set and Graded Mean Integration Representation Method. In section 3 we discuss the satisfaction level of demand maker which includes assumptions and notations. Section 4 provides a proof of the theorem on tardiness of jobs. In section 5 an algorithm is given which is applied to solve the numerical problems. Section 6 provides a numerical illustration. Concluding remarks are given in section 7.

Section-2

2.1 FUZZY CONCEPT AND GRADED MEAN INTEGRATION REPRESENTATION METHOD:

The significance of fuzzy variables is that they facilitate gradual transitions between states and consequently process a natural capability to express and deal with observation and measurement of uncertainties. Since fuzzy variables capture measurement uncertainties as part of experimental data they are closer to reality than crisp variables. Mathematics based on fuzzy sets has far greater expressive power than classical mathematics based on crisp sets; its usefulness depends critically on our capability to construct appropriate membership functions for given concept in various contexts. Chen S. H & Hsieh C.H. (1999) introduced graded mean integration representation method based on the integral value of graded mean h level of generalized fuzzy number for defuzzifying generalized fuzzy member. The generalized fuzzy number is defined as follows:

Suppose $\tilde{A}$ is a generalized fuzzy number. It is described as any fuzzy subset of the real line R, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions.

- $\mu_{\tilde{A}}$ (x) is a continuous mapping from R to [0,1],
- $\mu_{\tilde{A}}(x) = 0, -\infty < x \leq\cong a_{1,}$
- $\mu_{\tilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$,
- $\mu_{\tilde{A}}(x) = w_{A,} a_2 \leq x \leq a_3$,
- $\mu_{\tilde{A}}(x) = R(x)$ is strictly decreasing on $[a_3, a_4]$,
- $\mu_{\tilde{A}}(x) = 0, a_4 \leq x < \infty$,

Where $0< w_{A,} \leq 1$ and a_1, a_2, a_3 and a_4 are real numbers.

Generalized fuzzy numbers are denoted as $\tilde{A}= (a_1, a_2, a_3, a_4 ; w_A)_{LR}$. The graded mean h-level value of generalized fuzzy number $\tilde{A}= (a_1, a_2, a_3, a_4 ; w_A)_{LR}$ is given by $\frac{h}{2}\{L^{-1}(h) + R^{-1}(h)\}$. Then the graded Mean Integration Representation of P($\tilde{A}$) with grade w_A , where

$$P(\tilde{A}) = \frac{\int_0^{w_A} \frac{h}{2}\{L^{-1}(h)+R^{-1}(h)\}\, dh}{\int_0^{w_A} h\, dh},$$

Where $0 < h \leq w_A$ and $0< w_A \leq 1$.

Let $\tilde{A}$ be a trapezoidal fuzzy number and denoted as $\tilde{A} = (a_1, a_2, a_3, a_4)$. Then we can get the Graded mean Integration Representation of $\tilde{A}$ by the formula as:

$$P(\tilde{A}) = \frac{\int_0^1 \frac{h}{2}\{(a_1+a_4)+h(a_2-a_1-a_4+a_3)\}dh}{\int_0^1 h\, dh} = \frac{\boldsymbol{a_1+2a_2+2a_3+a_4}}{\boldsymbol{6}}$$

Section-3

3.1 SATISFACTION LEVEL: As demand maker wants that the job should be completed within time but in real situation, it has been observed that the output generally falls before or after the fixed time, which creates dissatisfaction to the demand maker. The satisfaction level depends on the time how much early or later it has been reached. The satisfaction level is defined as follows:

$$\mu_{i=} \begin{cases} 0 & ; \text{ if } x<d_{i1} \\ (x-d_{i1}) / (d_{i2}-d_{i1}) & ; \text{ if } d_{i1}<x<d_{i} \\ 1 & ; \text{ if } d_{i2}<x<d_{i3} \\ (d_{i4}-x)/ (d_{i4}-d_{i3}) & ; \text{ if } d_{i3}<x<d_{i4} \\ 0 & ; \text{ if } x>d_{i4} \end{cases}$$

3.2 ASSUMPTIONS AND NOTATIONS:

Assumptions:

- No job pre- emption is allowed.
- The machine can process one job at a time.
- All jobs are available at time zero.
- The machine set up time is negligible.
- Due date are fuzzy.

Notations:

A-<AHR> Average high ranking of fuzzy numbers (a, b, c, d) = (a +2b+2c+d)/6
A_{i1} - Average high ranking of the processing time of i^{th} job on 1^{st} machine.
A_{i2} - Average high ranking of the processing time of i^{th} job on 2^{nd} machine.
A_{i3} - Average high ranking of the processing time of i^{th} job on 3^{rd} machine.
A_{i4} - Average high ranking of the processing time of i^{th} job on 4^{th} machine.
P_i – Processing time of i^{th} job.
L – Total tardiness of all the jobs.
K- Constant factor for which total tardiness is optimized.
d_{ij} - Due time for i^{th} job and j^{th} fuzzy time
C_i – Completion time of i^{th} job.
μ_i – Degree of satisfaction level of i^{th} job for the demand maker.

Section-4

4.1 THEOREM ON TARDINESS OF THE JOBS: Let Pi be the process time of i^{th} job on both machines and KA_{i1} is the due time of the job. Then tardiness of the job is $L^2 = \sum(P_i - KA_{i1})^2$ where K is the factor depending on P_i & A_i is minimum for K $=\sum \frac{P_i A_{i1}}{A_{i1}^2}$.

PROOF: The result can be proved by simple calculus of Maxima and Minima. Total tardiness of the jobs is the time in which the jobs are idle i.e. $L^2 = \sum(P_i - KA_{i1})^2$ where i =1 to n.
On equating to zero the first derivative, we have
K $=\sum \frac{P_i A_{i1}}{A_{i1}^2}$. For the minimum value, second derivative should be positive.
$\frac{d^2L}{dK^2} = 2\sum A_{i1}^2 > 0$,being square .
L^2 is minimum for K $=\sum \frac{P_i A_{i1}}{A_{i1}^2}$
As due time depends upon the processing time of jobs on the first machine, so we desire to get the job early whose due time is early . Smaller the processing time early is the due time. Hence, the sequence of jobs can be made in non decreasing order. We summarise our work in the following Algorithm.

Section-5

ALGORITHM:

Step 1: Find <AHR> of the processing time of each job in fuzzy environment
(a, b, c, d) = (a +2b+2c+d)/6 on the basis of grade mean integration representation.
Step 2: Arrange the sequence in non decreasing order of A_{i1}'.
Step 3: Find the processing time P_i of each job i.
Step 4: Find the value of K by using the formula K $=\sum \frac{P_i A_{i1}}{A_{i1}^2}$
Step 5: Using the value of K, find due times of each job in fuzzy environment d_{ij} =K*processing time of i^{th} job in j^{th} fuzzy time.
Step 6: Find the completion time C_i of each job in fuzzy environment. We find the completion time C_i by using the formula $C_i = (C_{i1}+2C_{i2} + 2C_{i3} +C_{i4})/6$ on the basis of grade mean integration representation.
Step 7: Comparing completion time and due time of the jobs, we can find the satisfaction of demand maker.

Section-6

NUMERICAL EXAMPLE: Let there are 5 jobs & 4 machines whose processing time is given in trapezoidal fuzzy number as follows:

Job	M_1	M_2	M_3	M_4
1	(8,9, 10,12)	(5,6,7,8)	(2,3,4,5)	(5,7,9,10)
2	(15,16,17,18)	(10,14,15,16)	(9,10,11,12)	(6,8,10,11)
3	(2,6,8,10)	(8,10,11,12)	(6,7,8,10)	(10,11,12,13)
4	(5,6,7,8)	(15,16,18,20)	(12,13,14,15)	(3,5,7,8)
5	(9,12,13,15)	(6,8,9,10)	(8,9,10,12)	(10,12,13,14)

SOLUTION:

In order to solve the numerical problem, we proceed as per algorithm step by step.

Step1: Find <AHR> of the processing time of each job.

Job	1	2	3	4	5
A_1	58/6	99/6	40/6	39/6	74/6
A_2	39/6	100/6	62/6	103/6	50/6
A_3	21/6	63/6	46/6	81/6	58/6
A_4	47/6	53/6	69/6	35/6	74/6

Step 2: The Non decreasing sequence in which jobs are processed is 4-3-1-5-2.

Step 3: Processing time for each job is as follows:

Job			M_2		M_3		M_4	
	Time In	Time Out	Time In	Time Out	Time In	Time Out	Time In	Time Out
4	0	39/6	39/6	142/6	142/6	223/6	223/6	258/6
3	39/6	79/6	142/6	204/6	223/6	269/6	269/6	338/6
1	79/6	137/6	204/6	243/6	269/6	290/6	338/6	385/6
5	137/6	211/6	243/6	293/6	293/6	351/6	385/6	459/6
2	211/6	310/6	310/6	410/6	410/6	473/6	473/6	526/6

Hence P_1=385/6, P_2=526/6, P_3=338/6, P_4=258/6, P_5=459/6

Step 4: Using the formula $K = \sum \frac{P_i A_{i1}}{A_{i1}^2}$, we get K=6.06

Step 5: The list of due time in fuzzy environment is as follows:

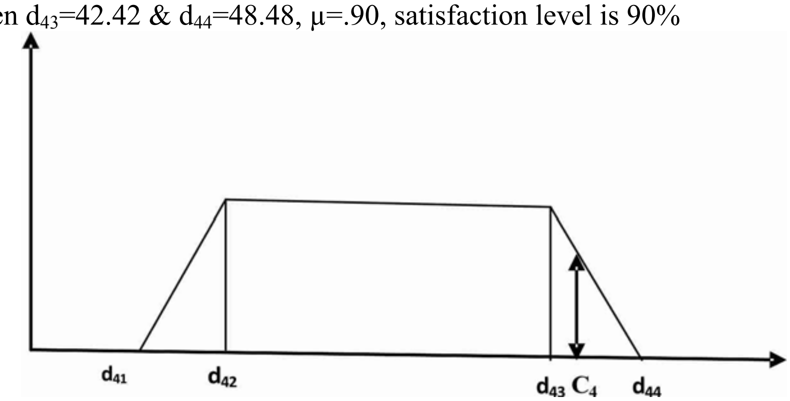
d_{11}=48.48	d_{12}=54.54	d_{13} =60.60	d_{14}=72.72
d_{21}= 90.90	d_{22}=96.96	d_{23}=103.02	d_{24}=109.0
d_{31}=12.12	d_{32}=36.36	d_{33}=48.48	d_{34}=60.60
d_{41} =30.30	d_{42} =36.36	d_{43}=42.42	d_{44}=48.48
d_{51}=54.54	d_{52}=72.72	d_{53}=78.78	d_{54}=90.90

Step 6: The completion time of jobs are as follows:

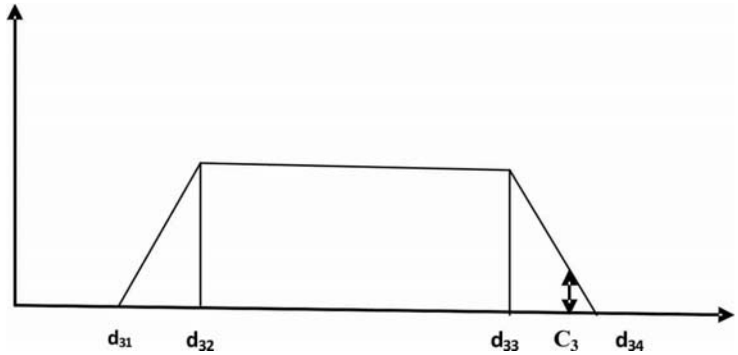
C_4=43.0, C_3=56.0, C_1=63.83, C_5=76.16 C_2=85.83.

Step 7: **Analytical Study:**

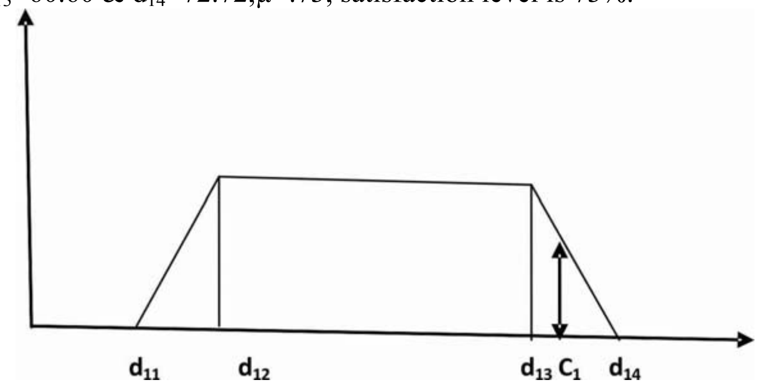
(a) C_4=43.0 lies between d_{43}=42.42 & d_{44}=48.48, μ=.90, satisfaction level is 90%



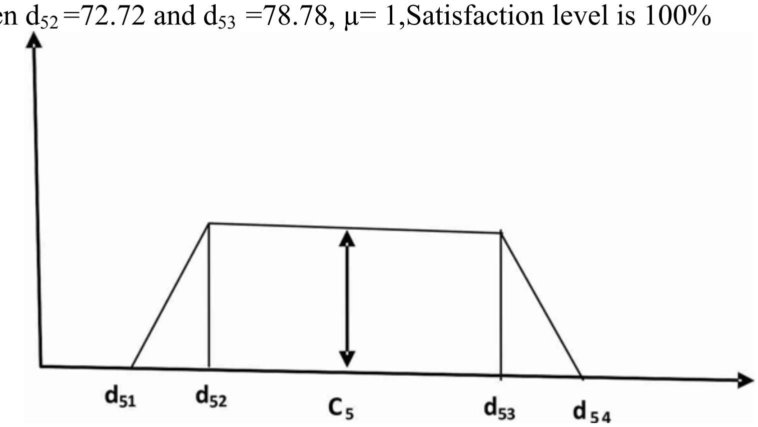
(b) C_3=56.0 lies between d_{33}=48.48 and d_{34}=60.60,μ= .37,Satisfaction level is 37%



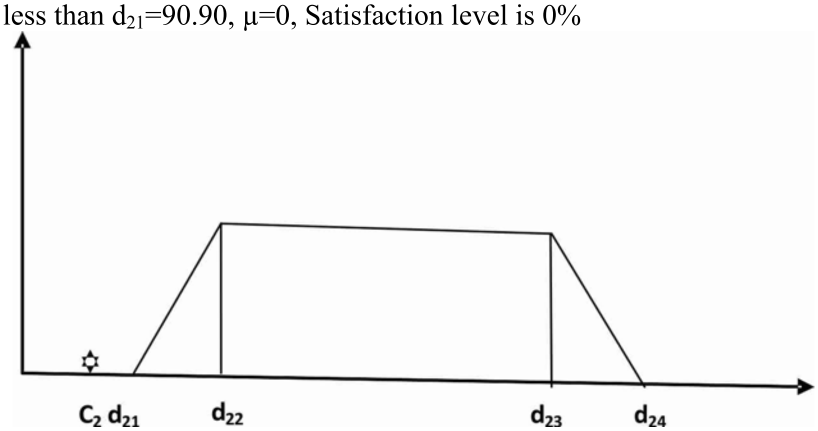
(c) C_1 =63.83lies between d_{13}=60.60 & d_{14}=72.72,μ=.73, satisfaction level is 73%.



(d) C_5=76.16 lies between d_{52} =72.72 and d_{53} =78.78, μ= 1,Satisfaction level is 100%



(e) C_2=85.83 which is less than d_{21}=90.90, μ=0, Satisfaction level is 0%



Section-7

CONCLUDING REMARKS AND FURTHER RESEARCH AREA:

In this study, a comparative analysis of completion time and due times of jobs has been calculated quantitatively and are presented graphically.

The study, has generalized the flow shop scheduling problem taking the processing time in fuzzy environment on m-machines. The contribution of our study is to achieve new results in scheduling theory with earliness and lateness penalties in a multi machines production under fuzzy environment. As processing time on m-machines are considered in fuzzy parameters, due time corresponding to the jobs are also fuzzy in nature.

From the numerical illustration, it is clear that if completion time of the job lies between 2^{nd} & 3^{rd} due time, the demand maker will be fully satisfied as the satisfaction level of demand maker is 100% for 5^{th} job and if completion time of the job lies either before the 1^{st} or after the 4^{th} due time, the customer will be quite unsatisfied as in case of job 2^{nd}.Based on the constraints imposed by due date, we have shown that the problem is decomposed into three sub problems as where the job comes before, comes after or lies between the due time. The paper is addressing multi machines problem with earliness and lateness penalties. The work can be extended in another direction by considering common due date for all jobs.

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A NOTE ON ORTHOGONAL IDEMPOTENTS IN FG, G IS AN ABELIAN GROUP

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ABSTRACT :

Explicit expressions for pair-wise orthogonal idempotents in FG, the semi simple group algebra of the abelian group G of order n over the finite field F = GF(q) of prime power order with $q = n\lambda + 1$ ***are obtained.***

AMS Mathematical Subject Classification (2000): 11T71, 11G25, 22D20.

Keywords : ***Group algebra, finite fields, orthogonal idempotents.***

1. INTRODUCTION

Let F = GF(q) be a field of prime power order q and let n be a positive integer which is relatively prime to q. The cyclic codes of length n over F can be viewed as ideals in either $F[x]/\langle x^n-1\rangle$ or as ideals in the group algebra FC_n, where C_n denotes a cyclic group of order n. The idempotent generators of minimal cyclic codes of length p^n(p^n = 2, 4 or p is an odd prime) where multiplicative order of q modulo p^n is $\emptyset(p^n)$ were completely described by Arora and Pruthi[1]. Also, idempotent generators of minimal cyclic codes of length $2p^n$ were discussed by Arora and Pruthi[2]. Recently Sheetal & Jagbir (9) obtained the explicit expression for the 4 (n+1) primitive idempotents in FG in which q is of the form 4k+1.

If G is an abeliangroup of order n then the ideals of the group algebra FG are called abelian codes. Ferraz and Milies[6] have found some minimal abelian codes of length p^n and $2p^n$ extending the results of Arora and Pruthi([1], [2]}. In this paper we describe the idempotents in FG, where G is an abelian group of order n and (charF, n) = 1. In section 2, we give expressions for some idempotents in FG, where G is an abelian Group of order n and F is a field of prime power order q and q = $n\lambda+1$ (Theorem 2.1). Theorem 2.4 gives the complete set of orthogonal idempotents in FG. In Corollary 2.5, we describe the orthogonal idempotents for the group algebra FG, G is abelian group of order p^n. In the section 3, we give example describing idempotents for the abelian group of order 10.

2. ORTHOGONAL IDEMPOTENTS IN FG, G IS AN ABELIAN GROUP

2.1 Theorem. Let G be an abelian group of order n and H is a subgroup of G of order m such that G/H is cyclic and G/H = <aH>. Let F be the field of order $n\lambda+1$ for some $\lambda > 0$, then

$$S_i = \left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{\frac{n}{m}-1}\alpha^{ij}a^j\right), \qquad 0 \le i \le \frac{n}{m}-1,$$

where α is $\left(\frac{n}{m}\right)^{th}$ root of unity in F, are orthogonal idempotents in FG.

Proof. Let G be an abelian group of order n and H be a subgroup of G of order m such that G/H is cyclic. Let G/H = <aH>.Here, $|G/H| = \frac{n}{m}$ = t(say).Also, $(aH)^t = H \Rightarrow a^tH = H \Rightarrow a^t \in H$. Consider a cyclic group G_1 of order t and let G_1 = <b>. Then $G_1 \cong G/H$ .The primitive idempotents of FG_1 are $e_i = \frac{1}{t}\sum_{j=0}^{t-1} \alpha^{ij} b^j$, $0 \le i \le t-1$, where α is t^{th} root of unity in F, that is, α is a solution of $x^t = 1$.Now consider the elements of FG given by

$$\xi_i = \frac{1}{t}\sum_{j=0}^{t-1} \alpha^{ij} a^j, \qquad 0 \le i \le t-1.$$

Then, $S_i = \left(\frac{1}{|H|}\sum_{h\in H} h\right)\xi_i = \left(\frac{1}{m}\sum_{h\in H} h\right)\left(\frac{1}{t}\sum_{j=0}^{t-1}\alpha^{ij}a^j\right) = \left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{t-1}\alpha^{ij}a^j\right)$, for $0 \le i \le t-1$, are orthogonal idempotents in FG.For $0 \le i \le t-1$, $S_i^2 = S_i$ as

$$S_i^2 = S_i S_i = \left(\left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{t-1}\alpha^{ij}a^j\right)\right)\left(\left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{t-1}\alpha^{ij}a^j\right)\right)$$

$$= \frac{1}{n^2}\left(\sum_{h\in H} h\right)^2 \left(1+\alpha^i a+\alpha^{2i}a^2+\ldots+\alpha^{(t-1)i}a^{t-1}\right)\left(1+\alpha^i a+\alpha^{2i}a^2+\ldots+\alpha^{(t-1)i}a^{t-1}\right)$$

$$= \frac{m}{n^2}\left\{t\left(\sum_{h\in H} h+\alpha^i a\sum_{h\in H} h+\alpha^{2i}a^2\sum_{h\in H} h+\ldots+\alpha^{(t-1)i}a^{t-1}\sum_{h\in H} h\right)\right\}$$

$$= \frac{1}{n}\left\{\sum_{h\in H} h\left(1+\alpha^i a+\alpha^{2i}a^2+\ldots+\alpha^{(t-1)i}a^{t-1}\right)\right\} = \left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{t-1}\alpha^{ij}a^j\right) = S_i$$

Also for i, j such that $i \ne j$, we let $i > j$, then

$$S_i S_j = \left(\left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{k=0}^{t-1}\alpha^{ik}a^k\right)\right)\left(\left(\frac{1}{n}\sum_{h\in H} h\right)\left(\sum_{l=0}^{t-1}\alpha^{jl}a^l\right)\right)$$

$$= \frac{m}{n^2}\left\{\sum_{h\in H} h+\alpha^i\sum_{h\in H} ha+\alpha^{2i}\sum_{h\in H} ha^2+\ldots+\alpha^{(t-1)i}\sum_{h\in H} ha^{t-1}+\alpha^j\sum_{h\in H} ha+\alpha^{i+j}\sum_{h\in H} ha^2\right.$$

$$\left.+\ldots+\alpha^{(t-1)i+j}\sum_{h\in H} ha^t+\ldots+\alpha^{(t-1)j}\sum_{h\in H} ha^{t-1}+\alpha^{i+(t-1)j}\sum_{h\in H} ha^t\ldots+\alpha^{(t-1)i+(t-1)j}\sum_{h\in H} ha^{2t-2}\right\}$$

Since $a^t \in H$, so we have $\sum_{g\in H} ga^t = \sum_{g\in H} g$ and $\sum_{g\in H} ga^{t+i} = \sum_{g\in H} ga^i$ for all $i \ge 0$

Using this the coefficient of $\frac{m}{n^2}\sum_{g\in H} ga^k$ in the above expression is

$$\left\{\alpha^{kj}+\alpha^{(k-1)j+i}+\alpha^{(k-2)j+2i}+\ldots+\alpha^{ki}\right\}+\left\{\alpha^{(k+1)i+(t-1)j}+\alpha^{(k+2)i+(t-2)j}+\ldots+\alpha^{(t-1)i+(k+1)j}\right\}$$

$$= \alpha^{kj}\left\{1+\alpha^{i-j}+\alpha^{2(i-j)}+\ldots+\alpha^{k(i-j)}\right\}+\alpha^{(k+1)i+(t-1)j}\left\{1+\alpha^{i-j}+\alpha^{2(i-j)}+\ldots+\alpha^{(t-k-2)(i-j)}\right\}$$

$$= \alpha^{kj}\left\{\frac{\alpha^{(k+1)(i-j)}-1}{\alpha^{(i-j)}-1}\right\}+\alpha^{(k+1)i+(t-1)j}\left\{\frac{\alpha^{(t-k-1)(i-j)}-1}{\alpha^{(i-j)}-1}\right\}$$

$$= \frac{1}{\alpha^{(i-j)}-1}\left\{\alpha^{kj}\left(\alpha^{(k+1)(i-j)}-1\right)+\alpha^{(k+1)i+(t-1)j}\left(\alpha^{(t-k-1)(i-j)}-1\right)\right\}$$

$$= \frac{1}{\alpha^{(i-j)}-1}\left\{\alpha^{(k+1)i-j}-\alpha^{kj}+\alpha^{ti+kj}-\alpha^{(k+1)i-(t-1)j}\right\}$$

$= \frac{1}{\alpha^{(i-j)}-1}\left\{\alpha^{(k+1)i-j} - \alpha^{kj} + \alpha^{kj} - \alpha^{(k+1)i-j}\right\}$ $[\because \alpha^t = 1]$

$= 0.$

Thus, $S_i S_j = 0$, and so $\{S_i\}$ $0 \le i \le t-1$, are orthogonal idempotents in FG.

2.2 Corollary. Let G be an abelian group of order p^n, where p is an odd prime, and F be a field of prime power order q with $q = p^n\lambda + 1$ for some $\lambda > 0$. If H is a subgroup of order p^{n-1} such that G/H = <aH> then $S_i = \left(\frac{1}{p^n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{p-1}\alpha^{ij}a^j\right)$, for $0 \le i \le p-1$, α is the p^{th} root of unity in F, are orthogonal idempotents in FG.

2.3 Corollary. Let G be an abelian group of order p^n, where p is an odd prime, and H is a subgroup of order p^m such that G/H is cyclic. Further suppose that G/H = <aH>. Let F be a field of prime power order q with $q = p^n\lambda + 1$ for some $\lambda > 0$. Then, $S_i = \left(\frac{1}{p^n}\sum_{h\in H} h\right)\left(\sum_{j=0}^{p^{n-m}-1}\alpha^{ij}a^j\right)$, for $0 \le i \le p^{n-m}-1$, where α is (p^{n-m})th roots of unity in F, are orthogonal idempotents in FG.

2.4 Theorem. Let G be an abelian group of order n and it has a sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_s \supset G_{s+1} = \langle e \rangle$$

such that $|G_i/G_{i+1}| = n_i$ and G_i/G_{i+1} is cyclic($0 \le i \le s$). Let F be a field of order q with q = nλ+1 for some λ > 0. Then, $\{e_{0,i_0}.e_{1,i_1}\ldots e_{s,i_s}\}$, $(0 \le i_j \le n_j - 1)(0 \le j \le s)$ where $G_j/G_{j+1} = \langle a_jG_{j+1}\rangle$ and $e_{j,i_j} = \left(\frac{1}{|G_j|}\sum_{g_{j+1}\in G_{j+1}} g_{j+1}\right)\left(\sum_{k=0}^{n_j-1}\alpha_j^{i_jk}a_j^k\right)$, α_j is n_j^{th} root of unity in F, are the orthogonal idempotents in $FG_j$for $0 \le j \le s$, is the complete set of orthogonal idempotents in FG.

Proof. Let G be an abelian group of order n and G_1 be the subgroup of G such that G/G_1 is cyclic. LetG/G_1 = <a_0G_1>. Then,

$$e_{0,i_0} = \left(\frac{1}{|G_1|}\sum_{g_1\in G_1} g_1\right)\left(\frac{1}{n_0}\sum_{j=0}^{n_0-1}\alpha_0^{i_0j}a_0^j\right) = \left(\frac{1}{n}\sum_{g_1\in G_1} g_1\right)\left(\sum_{j=0}^{n_0-1}\alpha_0^{i_0j}a_0^j\right),$$

for $0 \le i_0 \le n_0 - 1$, are orthogonal idempotents in FG, where α_0 is n_0^{th} root of unity in F.

Again, G_1 is an abelian group and G_2 is the subgroup of G_1 such that G_1/G_2 is cyclic. Let G_1/G_2 = <a_1G_2>.Then,

$$e_{1,i_1} = \left(\frac{1}{|G_2|}\sum_{g_2\in G_2} g_2\right)\left(\frac{1}{n_1}\sum_{j=0}^{n_1-1}\alpha_1^{i_1j}a_1^j\right) = \left(\frac{1}{|G_1|}\sum_{g_2\in G_2} g_2\right)\left(\sum_{j=1}^{n_1-1}\alpha_1^{i_1j}a_1^j\right),$$

for $0 \le i_1 \le n_1 - 1$, are orthogonalidempotents in FG, where α_1 is n_1^{th} root of unity in F.

Continuing in this way, we have G_{s-1} is an abelian group and G_s is the subgroup of G_{s-1} such that G_{s-1}/G_s is cyclic. LetG_{s-1}/G_s = <$a_{s-1}G_s$>.Then,

$$e_{s-1,i_{s-1}} = \left(\frac{1}{|G_s|}\sum_{g_s\in G_s} g_s\right)\left(\frac{1}{n_{s-1}}\sum_{j=0}^{n_{s-1}-1}\alpha_{s-1}^{i_{s-1}j}a_{s-1}^j\right) = \left(\frac{1}{|G_{s-1}|}\sum_{g_s\in G_s} g_s\right)\left(\sum_{j=0}^{n_{s-1}-1}\alpha_{s-1}^{i_{s-1}j}a_{s-1}^j\right),$$

for $0 \le i_{s-1} \le n_{s-1} - 1$, are orthogonalidempotents in FG_{s-1}, where α_{s-1} is n_{s-1}^{th} root of unity in F.

Further, since $G_s / G_{s+1} \cong G_s$ is cyclic group of order n_s. Let G_s = <a_s>. Then, orthogonal idempotents in FG_s are

$$e_{s,i_s} = \left(\frac{1}{|G_s|}\sum_{j=0}^{n_s-1}\alpha_s^{i_sj}a_s^j\right),$$

for $0 \le i_s \le n_s - 1,$ where α_s is n_s^{th} root of unity in F.

Now,

$$e_i = e_{i_s n_{s-1} n_{s-2} \ldots n_0 + i_{s-1} n_{s-2} n_{s-3} \ldots n_0 + \ldots + i_1 n_0 + i_0} = e_{0,i_0} . e_{1,i_1} \ldots e_{s,i_s}$$

where $i = i_s n_{s-1} n_{s-2} \ldots n_0 + i_{s-1} n_{s-2} n_{s-3} \ldots n_0 + \ldots + i_1 n_0 + i_0$, are pairwise orthogonal idempotents in FG, as

$$e_i^2 = \left(e_{i_s n_{s-1} n_{s-2} \ldots n_0 + i_{s-1} n_{s-2} n_{s-3} \ldots n_0 + \ldots + i_1 n_0 + i_0}\right)^2 = \left(e_{0,i_0} . e_{1,i_1} \ldots e_{s,i_s}\right)^2 = \left(e_{0,i_0}\right)^2 \left(e_{1,i_1}\right)^2 \ldots \left(e_{s,i_s}\right)^2$$

$$= e_{0,i_0} . e_{1,i_1} \ldots e_{s,i_s} = e_{i_s n_{s-1} n_{s-2} \ldots n_0 + i_{s-1} n_{s-2} n_{s-3} \ldots n_0 + \ldots + i_1 n_0 + i_0} = e_i$$

For $i \neq j$, the representation of i and j are as follows

$$i = i_s n_{s-1} n_{s-2} \ldots n_0 + i_{s-1} n_{s-2} n_{s-3} \ldots n_0 + \ldots + i_1 n_0 + i_0$$

$$j = j_s n_{s-1} n_{s-2} \ldots n_0 + j_{s-1} n_{s-2} n_{s-3} \ldots n_0 + \ldots + j_1 n_0 + j_0$$

and they must differ at atleast one indces, say k^{th}, that is $i_k \neq j_k$, then

$$e_{k,i_k} . e_{k,j_k} = 0.$$

Thus, $e_i . e_j = 0.$

2.5 Corollary. Let G be an abelian group of order p^n and F be a field of order q with $q = p^n \lambda + 1$ for some $\lambda > 0$. Then, by considering a sequence of subgroups of G such that it has a sequence of subgroups $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_s = < e >$ with the property that G_{i+1} is subgroup of Gof smallest order $p^{n_{i+1}}$ (say), such that G_i/G_{i+1} is cyclic. Then,

$$\left\{e_{0,i_0} . e_{1,i_1} \ldots e_{s-1,i_{s-1}}\right\}, \qquad \left(0 \le i_j \le \frac{p^{n_j}}{p^{n_{j+1}}} - 1\right)\left(0 \le j \le s-1\right)$$

where $G_j/G_{j+1} = <a_j G_{j+1}>$ and $e_{j,i_j} = \left(\frac{1}{p^{n_j}} \sum_{g_{j+1} \in G_{j+1}} g_{j+1}\right)\left(\sum_{k=0}^{p^{n_j - n_{j+1}} - 1} \alpha_j^{i_j k} a_j^k\right)$, α_j is $\left(p^{n_j - n_{j+1}}\right)^{th}$ root of unity in F, are the orthogonalidempotents in FG_j for $0 \le j \le s-1$, are orthogonal idempotents in FG.

Proof. Let G be the abelian group of order p^n. If G is cyclic then the primitive idempotentsof FG are described by Arora and pruthi[1]. If G is not cyclic then assume that G_1 be a subgroup of G of order p^m with m smallest such that G/G_1 is cyclic. Further, assume that

$G/G_1 = <a G_1>$.

Then, $|G/G_1| = p^{n-m}$ and

$$e_{0,i_0} = \frac{1}{p^n}\left(\sum_{g \in G_1} g\right)\left(\sum_{j=0}^{p^{n-m}-1} \alpha^{i_0 j} a^j\right),$$

for $0 \le i_0 \le p^{n-m} - 1,$ where α is the $(p^{n-m})^{th}$ root of unity in F, are orthogonal idempotents in FG.

We discuss different case to complete the solution:

Case I:- G_1 is cyclic.Let $G_1 = <b>$.Then, the primitive idempotents of FG_1 are given by $e_{1,i_1} = \frac{1}{p^m}\left(\sum_{j=0}^{p^m - 1} \beta^{i_1 j} b^j\right)$, for $0 \le i_1 \le p^m - 1,$ where β is the $(p^m)^{th}$ root of unity in F.In this case $e_{i_0 + p^{n-m} i_1} = e_{0,i_0} . e_{1,i_1}$ for $0 \le i_0 \le p^{n-m} - 1,$ $0 \le i_1 \le p^m - 1$,are orthogonal idempotents in FG.

Case II:-G_1 is not cyclic.Let G_2 be a subgroup of G_1 of order p^t such that G_1/G_2 is cyclic. Let $G_1/G_2 = \langle a_1G_2\rangle$.Here, $|G_1/G_2| = p^{m-t}$.Then, $e_{2,i_2} = \frac{1}{p^m}\left(\sum_{h\in G_2} h\right)\left(\sum_{j=0}^{p^{m-t}-1} \alpha_1^{i_2 j} b^j\right)$, for $0 \le i_2 \le p^{m-t}-1$, where α_1 is the $(p^{m-t})^{th}$ root of unity in F, are orthogonalidempotents in FG_1.If G_2 is cyclic, then as in case I, we find the orthogonal idempotents in FG_2 and then orthogonalidempotents of FG are of the form $e_{0,i_0}e_{1,i_1}e_{2,i_2}$, for $0 \le i_0 \le p^{n-m}-1,\ 0 \le i_1 \le p^{m-t}-1,$ $0 \le i_2 \le p^t - 1$, are orthogonal idempotents in FG.

If G_2 is not cyclic, then continuing as in case II, we find a sequence of subgroups
$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_s = \langle e\rangle$
such that G/G_1, G_1/G_2, G_2/G_3, …, G_{s-1}/G_s, are all cyclic then the products of all the orthogonalidempotents so obtained as above are the orthogonalidempotents for FG.
(The proof of following is on similar lines as that of Corollary 2.5 with slight modification).

2.6 Corollary. Let G be an abelian group of order p^n and it has a sequence of subgroups $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_{n-1} \supset G_n = \langle e\rangle$ such that $|G_i/G_{i+1}| = p$, $0 \le i \le n-1$. Let F be a field of order q with $q = p^n\lambda + 1$ for some $\lambda > 0$. Then, $\{e_{0,i_0}.e_{1,i_1}\ldots e_{n-1,i_{n-1}}\}$, $(0 \le i_j \le p)(0 \le j \le n-1)$ where $G_j/G_{j+1} = \langle a_jG_{j+1}\rangle$ and $e_{j,i_j} = \left(\frac{1}{p^{n-j}}\sum_{g_{j+1}\in G_{j+1}} g_{j+1}\right)\left(\sum_{k=0}^{p-1}\alpha^{i_j k} a_j^k\right)$, α is p^{th} root of unity in F, are the orthogonalidempotents in $FG_j$for $0 \le j \le n-1$, are orthogonalidempotents in FG.

2.7 Theorem. Let G be an abelian group of order n and it can be expressed as $G = G_0 \text{ x } G_1 \text{ x } G_2 \text{ x} \ldots \text{x } G_s$ such that $|G_i| = n_i$ and G_i is cyclic($0 \le i \le s$). Let F be a field of order q with $q = n\lambda+1$ for some $\lambda > 0$. Then, $\{e_{0,i_0}.e_{1,i_1}\ldots e_{s,i_s}\}$, $(0 \le i_j \le n_j - 1)(0 \le j \le s)$ where $G_j = \langle a_j\rangle$ and $e_{j,i_j} = \frac{1}{n_i}\sum_{k=0}^{n_j-1}\alpha_j^{i_j k} a_j^k$, α_j is n_j^{th} root of unity in F, are the pairwise orthogonal idempotents in $FG_j$for $0 \le j \le s$, are pairwise orthogonal idempotents in FG whose sum is 1.

Proof of this can easily be proved on the similar lines as that of Theorem 3.2.4 with slight modification and the sum of all these idempotents is 1, can be deduced easily using the fact that if α_j is n_j^{th} root of unity in F, then $1+\alpha_j+\alpha_j^2+\ldots+\alpha_j^{n_j-1} = 0.$

3 EXAMPLE

In this example we describe the orthogonal idempotents for FG, where G is an abelian group of order 10.

3.1 Example. Let G be an abelian group of group 10. Then, G will have a subgroup H of order 2 and|G/H| = 5. Let F be a field of order 11.Let $G/H = \langle aH\rangle$. Then,

$$e_{0,0} = \left(\frac{1}{10}\sum_{g\in H} g\right)\left(1+a+a^2+a^3+a^4\right), \qquad e_{0,1} = \left(\frac{1}{10}\sum_{g\in H} g\right)\left(1+\alpha a+\alpha^2a^2+\alpha^3a^3+\alpha^4a^4\right),$$

$$e_{0,2} = \left(\frac{1}{10}\sum_{g\in H} g\right)\left(1+\alpha^2a+\alpha^4a^2+\alpha a^3+\alpha^3a^4\right), \; e_{0,3} = \left(\frac{1}{10}\sum_{g\in H} g\right)\left(1+\alpha^3a+\alpha a^2+\alpha^4a^3+\alpha^2a^4\right),$$

$$e_{0,4} = \left(\frac{1}{10}\sum_{g\in H} g\right)\left(1+\alpha^4a+\alpha^3a^2+\alpha^2a^3+\alpha a^4\right)$$

areorthogonal idempotents in FG, where α is solution of $x^5 = 1$ in F.

Also, |H| = 2. Let H = < b >. Then, orthogonal idempotents of FH are $e_{1,0} = \frac{1+b}{2}$, $e_{1,1} = \frac{1-b}{2}$. Then,

$$e_0 = e_{0,0}e_{1,0} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+a+a^2+a^3+a^4+b+ab+a^2b+a^3b+a^4b\right)$$

$$e_1 = e_{0,1}e_{1,0} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha a+\alpha^2a^2+\alpha^3a^3+\alpha^4a^4+b+\alpha ab+\alpha^2a^2b+\alpha^3a^3b+\alpha^4a^4b\right)$$

$$e_2 = e_{0,2}e_{1,0} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha^2a+a^2+\alpha^2a^3+\alpha^3a^4+b+\alpha^2ab+a^2b+\alpha^2a^3b+\alpha^3a^4b\right)$$

$$e_3 = e_{0,3}e_{1,0} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha^3a+\alpha^2a^2+\alpha a^3+\alpha a^4+b+\alpha^3ab+\alpha^2a^2b+\alpha a^3b+\alpha a^4b\right)$$

$$e_4 = e_{0,4}e_{1,0} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha^4a+\alpha^3a^2+\alpha^2a^3+\alpha a^4+b+\alpha^4ab+\alpha^3a^2b+\alpha^2a^3b+\alpha a^4b\right)$$

$$e_5 = e_{0,0}e_{1,1} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+a+a^2+a^3+a^4-b-ab-a^2b-a^3b-a^4b\right)$$

$$e_6 = e_{0,1}e_{1,1} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha a+\alpha^2a^2+\alpha^3a^3+\alpha^4a^4-b-\alpha ab-\alpha^2a^2b-\alpha^3a^3b-\alpha^4a^4b\right)$$

$$e_7 = e_{0,2}e_{1,1} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha^2a+a^2+\alpha^2a^3+\alpha^3a^4-b-\alpha^2ab-a^2b-\alpha^2a^3b-\alpha^3a^4b\right)$$

$$e_8 = e_{0,3}e_{1,1} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha^3a+\alpha^2a^2+\alpha a^3+\alpha a^4-b-\alpha^3ab-\alpha^2a^2b-\alpha a^3b-\alpha a^4b\right)$$

$$e_9 = e_{0,4}e_{1,1} = \left(\frac{1}{20}\sum_{g\in H} g\right)\left(1+\alpha^4a+\alpha^3a^2+\alpha^2a^3+\alpha a^4-b-\alpha^4ab-\alpha^3a^2b-\alpha^2a^3b-\alpha a^4b\right)$$

are orthogonal idempotents in FG.

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TESTING OF PROPER RANDOMNESS OF THE NUMBERS GENERATED BY KENDALL AND B. BABINGTON SMITH: t-TEST

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ABSTRACT :

Proper randomness of the numbers generated by Kendall and B. Babington Smith has been examined.by B.K. Sarmah and D. Chakrabarty in Dec.. 2014, IJESRT Vol-3, issue 12, by applying the Chi-square test for testing the significance of difference between observed frequency of each of the digit in the table and the corresponding theoretical (expected) frequency.

In this paper, the randomness of the digits have been tested by applying t-test for amount of deviation of the observed number of occurrences and the theoretical(expected) number of occurrences of the respective digits and hence the numbers. The test shows that the numbers generated by Kendall and B. Babington Smith deviated significantly in most the observations from proper randomness.

Keywords: ***Random number generated by Kendall and B. Babington Smith, student t-test, testing of randomness.***

1. INTRODUCTION

Drawing of random sample has been found to be vital or basic necessity in most of the researches and investigations especially of applied sciences. The convenient practical method of selecting a random sample consists of the use of Table of Random Numbers. Existing tables of random numbers, used commonly, are the ones due to Fisher and Yates (Constructed in 1938), Kendall and B. Babington Smith (Constructed in 1927), Kendall and Babington Smith (Constructed in 1939) and Rand Corporation (constructed in 1955)

The random number tables have been subjected to various statistical tests of randomness. These tests have limitations to decide on proper randomness of the numbers occurring in the corresponding tables. As a Consequence it is not guaranteed that the numbers in each of these tables are properly random. This leads to think of testing the proper randomness of the numbers in this tables. In the present study, an attempt has been made to test this. The study, here, has been made on the testing of randomness of the table of numbers constructed by Kendall and B. Babington Smith only.

By the existing statistical methods, it is only possible to know whether the randomness of the numbers of a table is proper. It is only possible to know whether the deviation of the degree of its randomness is significant.

In order to test the proper randomness of the random numbers table constructed by Kendall and B. Babington Smith t-test has been applied.

2. MATERIALS AND METHODS:

Kendall and B. Babington Smith random number table consists of a total of 100000 digits arranged in 25000 four digited numbers Here a sample of 47700 digits are considered out of 100000 digits for testing randomness.

To know whether the number in random numbers table of Kendall and B. Babington Smith are proper or not student's t-test for amount of deviation is applied.

Let 'd' be the variable denoting the measure of the deviation (amount of deviation) of the observed number of occurrences of the respective digit.

Suppose, $d_i (i=0,1\ldots.9)$ are independent observed values of the deviation variables.

If the table of number is random then di=0, for all i, in the ideal situation . However, due to chance error, di may assume non zero value.

Thus the values of di's are due to chance error but not due to any assignable error if the table is random.

The chance variables are i.i.d. N.(o,σ) variables.

Thus testing of randomness is equivalent to testing of the hypothesis H_o

That E(d_i) = 0, for all i,

Let us consider the statistic t for testing H_o

i.e. $t = \frac{\overline{d} - E(\overline{d})}{S.E.(\overline{d})} \sim t_{n-1}$

where,

$\overline{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$

We have,

$E(\overline{d}) = \frac{1}{n} \sum E(d_i) = 0$

When H_o is true

Also,

$var(\overline{d}) = \frac{\sigma^2}{n}$, σ^2 is unknown

However unbiased estimate of σ^2 is

$s^2 = \frac{1}{n-1} \sum (d_i - \overline{d})^2$

$= \frac{1}{n-1}[\Sigma d_i^2 - \frac{(\Sigma d_i)^2}{n}]$

Which implies unbiased estimate of

$var(\overline{d}) = \frac{s^2}{n}$

$and\ S.D.(\overline{d}) = \frac{s}{\sqrt{n}}$

Therefore statistic t for testing H_o becomes

$t = \frac{\overline{d}}{s/\sqrt{n}}$ when H_o is true and this t follows student's t distribution with (n-1) d.f.

3. STEPS IN THE METHOD:

In order to test the proper randomness of the numbers of Kendall and B. Babington Smith table one is required to proceed with the following steps:

Step1: In the first step, observe the occurrences of the digits 0 to 9 for first 2000 trails, second 2000 trials ……….. … up to 23rd 2000 trails and lastly for 1700 trails as shown in the table.

Step2: In the second step, compute the theoretical expected frequencies. This is done by dividing trails i.e. 1st 2000 ………… and 23rd 2000 and last 1700 trails by 10 assuming that the digit 0 to 9 occurs equal number of times.

Step 3: In the third step, compute the amount of deviation of observed occurrences of digits and expected occurrences of digits.

Step 4: In the fourth step, compute the value of student's t for each of the trails.

Step 5: Compare the value of t-statistic with corresponding theoretical values.

Step 6: Draw conclusion as per the result obtained in step 5.

4. RESULTS AND DISCUSSION:

The results obtained on operating the steps (Nos. 1 to 5) on the random numbers table constructed by Kendall and B. Babington Smith have been observed. It is observed from the table that occurrences of digit 0 to 9 are not equal.

5. CONCLUSION:

From the table prepared for observed frequency of occurrence of digits along with respective expected frequency (shown in bracket) it is observed that the calculated value of t is significant in all most of the cases except 5th 2000 trails. That is calculated value of 't' have been found to be significant on comparing them with the corresponding theoretical values for most of the cases (trails).

Hence it may be concluded that the table of numbers constructed by Kendall and B. Babington Smith deviates significantly from proper randomness.

Therefore Kendall and B. Babington Smith Random Numbers Table Cannot be treated as properly random, as per results obtained by applying t-test.

6. TABLE

Observed frequency of occurrence of digits along with the respective expected frequency (Shown in bracket), amount of deviation (di) and the values of students 't' statistic from Kendall and B. Babington Smith.

Digits	0	1	2	3	4	5	6	7	8	9	Value of t
1st 2000	1 199 (200)	11 189 (200)	4 204 (200)	14 214 (200)	10 190 (200)	1 199 (200)	4 204 (200)	5 205 (200)	1 199 (200)	3 197 (200)	3.50
2nd 2000	17 183 (200)	6 206 (200)	3 203 (200)	17 183 (200)	11 189 (200)	10 210 (200)	3 203 (200)	4 196 (200)	4 196 (200)	31 231 (200)	4.38
3rd 2000	3 197 (200)	3 197 (200)	7 207 (200)	2 202 (200)	0 200 (200)	3 203 (200)	11 189 (200)	11 211 (200)	15 185 (200)	9 209 (200)	3.59
4th 2000	10 190 (200)	5 195 (200)	5 195 (200)	13 213 (200)	18 182 (200)	2 202 (200)	2 198 (200)	18 218 (200)	1 201 (200)	6 206 (200)	3.64
5th 2000	12 212 (200)	14 186 (200)	39 239 (200)	6 194 (200)	8 192 (200)	4 196 (200)	8 192 (200)	2 202 (200)	10 190 (200)	3 197 (200)	3.12*
6th 2000	12 212 (200)	24 176 (200)	4 196 (200)	17 217 (200)	16 216 (200)	20 180 (200)	2 202 (200)	6 194 (200)	9 209 (200)	2 198 (200)	4.85
7th 2000	3 203 (200)	1 199 (200)	16 184 (200)	3 197 (200)	8 208 (200)	2 198 (200)	14 214 (200)	12 212 (200)	9 191 (200)	6 194 (200)	4.06
8th 2000	25 175 (200)	23 223 (200)	15 185 (200)	3 203 (200)	3 203 (200)	8 192 (200)	6 206 (200)	0 200 (200)	13 213 (200)	0 200 (200)	3.57
9th 2000	4 204 (200)	11 189 (200)	24 176 (200)	10 190 (200)	39 239 (200)	6 206 (200)	2 202 (200)	9 191 (200)	22 178 (200)	25 225 (200)	3.54
10th 2000	1 201 (200)	13 213 (200)	4 204 (200)	12 212 (200)	25 175 (200)	16 184 (200)	6 206 (200)	5 195 (200)	15 215 (200)	5 195 (200)	4.30
11th 2000	6 194 (200)	12 212 (200)	3 203 (200)	8 208 (200)	9 209 (200)	15 185 (200)	4 196 (200)	21 179 (200)	9 209 (200)	5 205 (200)	5.12
12th 2000	5 205 (200)	12 188 (200)	1 201 (200)	9 209 (200)	11 211 (200)	7 193 (200)	23 223 (200)	4 204 (200)	18 182 (200)	16 184(200)	4.30
13th 2000	32 232 (200)	17 217 (200)	1 201 (200)	11 211 (200)	13 211 (200)	3 197 (200)	30 170 (200)	11 211 (200)	6 194 (200)	20 180 (200)	3.77
14th 2000	8 208 (200)	10 190 (200)	19 181 (200)	25 175 (200)	8 192 (200)	15 215 (200)	2 198 (200)	2 198 (200)	13 213 (200)	30 230	4.48
15 th 2000	10 190 (200)	7 193 (200)	12 188 (200)	8 208 (200)	29 171 (200)	13 213 (200)	16 216 (200)	4 204 (200)	7 207 (200)	10 210 (200)	4.77
16th 2000	5 195 (200)	3 197 (200)	14 186 (200)	6 206 (200)	15 215 (200)	7 207 (200)	3 203 (200)	14 186 (200)	20 180 (200)	25 225 (200)	4.71
17th 2000	7 207 (200)	1 199 (200)	9 191 (200)	5 205 (200)	3 197 (200)	5 195 (200)	13 213 (200)	3 203 (200)	6 194 (200)	4 196 (200)	4.82
18th 2000	3 203 (200)	23 177 (200)	17 217 (200)	10 190 (200)	11 189 (200)	2 202 (200)	1 199 (200)	18 218 (200)	3 197 (200)	8 208 (200)	3.62
19th 2000	20 220 (200)	1 199 (200)	21 179 (200)	4 204 (200)	1 201 (200)	15 215 (200)	8 192 (200)	6 194 (200)	10 190 (200)	6 206 (200)	3.78
20th 2000	14 214 (200)	9 191 (200)	23 177 (200)	2 202 (200)	3 197 (200)	17 217 (200)	6 206 (200)	2 202 (200)	8 192 (200)	2 202 (200)	3.83
21st 2000	4 204 (200)	4 196 (200)	21 221 (200)	11 211 (200)	10 190 (200)	14 186 (200)	6 206 (200)	15 185 (200)	16 184 (200)	17 217 (200)	5.76
22nd 2000	8 192 (200)	1 201 (200)	20 180 (200)	7 187 (200)	20 220 (200)	24 224 (200)	13 187 (200)	2 202 (200)	3 197 (200)	10 210 (200)	3.76
23rd 2000	1 201 (200)	7 207 (200)	19 181 (200)	23 223 (200)	8 208 (200)	34 166 (200)	17 217 (200)	12 188 (200)	2 202 (200)	7 207 (200)	3.83
Last 1700	6 164 (170)	16 186 (170)	11 159 (170)	22 192 (170)	6 176 (170)	8 162 (170)	17 153 (170)	18 152 (170)	14 184 (170)	2 172 (170)	6.91

*Indicates the values which are less than theoretical values of t 0.01,8 =3.36

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ON THE DIOPHANTINE EQUATION $a\prod x_i = b\sum x_i + k$

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ABSTRACT :
In this paper, the Diophantine equation of the form $a\prod x_i = b\sum x_i + k$, where a, b are positive integer and k is an integer, has been discussed for positive integral as well as integral solutions.
Keywords: ***Diophantine equation and positive integral solution.***
Subject Classification: 11D45.

1 INTRODUCTION: Several authors discussed the Diophantine equation $\prod x_i = \sum x_i$ for restriction on x_i given by $x_1 \geq x_2 \geq \cdots \geq 1$. **R.K. Guy** (2004) in Some Unsolved Problems in Number Theory (problem D24) mentioned that for $n \geq 2$ the Diophantine equation

$$\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i, (x_1 \geq x_1 \geq \cdots x_n \geq 1), \quad \ldots(1)$$

has the solution $x_1 = n, x_2 = 2, x_3 = x_4 = \cdots x_n = 1$. This is the only solution in positive integers when $n = 2,3,4,6,24,114,174$ and 444. **Misiuewicz** (1966) verified the result for $n \leq 10^3$ and **Brown** (1984) for $n \leq 5.10^4$. It seems that **Trost** (1956) was first to pose the problem of solving $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i = 1$ in rationals. **Schinzel** (1961) (unsolved problems p238) conjectured that there is a k such that

$$\prod_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i = n, (x_1 \geq x_1 \geq \cdots x_k$$

has a solution for every sufficient large n. **Viola** (1993) discussed the Diophantine equation

$$\prod_{i=0}^{k} x_i - \sum_{i=0}^{k} x_i = n$$

and

$$\sum_{0}^{k} \frac{1}{x_i} = \frac{k}{n}.$$

Weingartner (2012) discussed the Diophantine equation $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$. For $n \geq 2$, he assumed $f(n)$ the number of positive solution to the equation

$$\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$$

with $x_1 \geq x_1 \geq \cdots x_n \geq 1$ and established an asymptotic formula for the average order of $f(n)$.

Pramod etal (2012) obtained some integral solutions of Diophantine equations $x^3+y^3+z^3 = 3xyz+(x+y+z)\, w^n$ for n = 2, 3 & 4.

In this paper, the Diophantine equation of the form $a\prod x_i = b\sum x_i + k$, where a, b are positive integer and k is an integer, will be discussed for positive integral as well as integral solutions.

2. ANALYSIS: (i) Diophantine equation of the form $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$ with the condition $x_1 > x_2 > x_3 > \cdots > x_n$:

This Diophantine equation has only one positive integral solution given by $(x_1, x_2, x_3) = (3,2,1)$ as $3.2.1 = 3 + 2 + 1$.

The above Diophantine equation has only two integral solutions given by $(x_1, x_2, x_3) = (3,2,1)$ and $(-1,-2,-3)$ as $3.2.1 = 3 + 2 + 1$ and $(-1).(-2).(-3) = (-1) + (-2) + (-3)$.

Thus we have the following theorem:

Theorem 1: The Diophantine equation of the form $\prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$ with the condition $x_1 > x_2 > x_3 > \cdots > x_n$ has only two integral solutions given by $(x_1, x_2, x_3) = (3,2,1)$ and $(-1,-2,-3)$.

(ii) Diophantine equation of the form $2\prod_{i=1}^{n} x_i = 3\sum_{i=1}^{n} x_i - 3$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 2$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = 3, x_3 = 1, \ldots, x_n = 1$ as

$$2n.3.1 \ldots 1 = 3(n + 3 + 1 \ldots + 1) - 3.$$

(iii) Diophantine equation of the form $\prod_{i=1}^{n} x_i = 2\sum_{i=1}^{n} x_i - 4$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 2$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = m, x_3 = 1, \ldots, x_n = 1$ as

$$n.m.1 \ldots 1 = 2(n + m + 1 \ldots + 1) - 4.$$

(iv) In general, the Diophantine equation of the form

$2\prod_{i=1}^{n} x_i = m\sum_{i=1}^{n} x_i - m(m-2)$with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 2$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = m, x_3 = 1, \ldots, x_n = 1$ as

$$2n.m.1 \ldots 1 = m2(n + 4 + 1 \ldots + 1) - m(m-2).$$

The above Diophantine equation can also be written as

$2\prod_{i=1}^{n} x_i = m(\sum_{i=1}^{n} x_i - (m-2))$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 2$.

Thus we have the following theorem:

Theorem 2: The Diophantine equation of the form $2\prod_{i=1}^{n} x_i = m\sum_{i=1}^{n} x_i - m(m-2)$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 2$ has only one positive integral solution given by $x_1 = n, x_2 = m,\ x_3 = 1, \ldots,$ $x_n = 1$.

(v) Diophantine equation of the form $\prod_{i=1}^{n} x_i = 2(\sum_{i=1}^{n} x_i - 1)$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 3$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = 2,\ x_3 = 2, x_4 = 1 \ldots,$ $x_n = 1$ as

$$\prod_{i=1}^{n} x_i = 2n.2.1 \dots 1 = 4n, \quad \dots(2)$$

$$\sum_{i=1}^{n} x_i = n + 2 + 2 + 1 \dots + 1$$

$$= 2n + 1. \quad \dots(3)$$

From (2) and (3), we have

$$\prod_{i=1}^{n} x_i = 2\left(\sum_{i=1}^{n} x_i - 1\right).$$

(vi) Diophantine equation of the form $\prod_{i=1}^{n} x_i = 3(\sum_{i=1}^{n} x_i - 2)$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 3$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = 2,\ x_3 = 3, x_4 = 1 \dots,$ $x_n = 1$ as

$$\prod_{i=1}^{n} x_i = 2n.3.1 \dots 1 = 6n, \quad \dots(4)$$

$$\sum_{i=1}^{n} x_i = n + 2 + 3 + 1 \dots + 1$$

$$= 2n + 2. \quad \dots(5)$$

From (4) and (5), we have

$$\prod_{i=1}^{n} x_i = 3\left(\sum_{i=1}^{n} x_i - 2\right).$$

(vii) Diophantine equation of the form $\prod_{i=1}^{n} x_i = 4(\sum_{i=1}^{n} x_i - 3)$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 3$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = 2,\ x_3 = 4, x_4 = 1 \dots,$ $x_n = 1$ as

$$\prod_{i=1}^{n} x_i = 2n.4.1 \dots 1 = 8n, \quad \dots(6)$$

$$\sum_{i=1}^{n} x_i = n + 2 + 4 + 1 \dots + 1$$

$$= 2n + 3. \quad \dots(7)$$

From (6) and (7), we have

$$\prod_{i=1}^{n} x_i = 4\left(\sum_{i=1}^{n} x_i - 3\right).$$

(viii) Diophantine equation of the form $2\prod_{i=1}^{n} x_i = m_1.m_2(\sum_{i=1}^{n} x_i - m_1 - m_2 + 3)$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 3$:

This Diophantine equation has only one positive integral solution given by $x_1 = n, x_2 = m_1, x_3 = m_2, x_4 = 1 \dots,$ $x_n = 1$ as

$$\prod_{i=1}^{n} x_i = n.m_1.m_2.1 \dots 1 = m_1.m_2.n, \qquad \dots(8)$$

$$\sum_{i=1}^{n} x_i = n + m_1 + m_2 + 1 \dots + 1$$

$$= 2n + m_1 + m_2 - 3. \qquad \dots(9)$$

From (8) and (9), we have $2\prod_{i=1}^{n} x_i = m_1.m_2(\sum_{i=1}^{n} x_i - m_1 - m_2 + 3)$.

Theorem 3: The Diophantine equation of the form $2\prod_{i=1}^{n} x_i = m_1.m_2(\sum_{i=1}^{n} x_i - m_1 - m_2 + 3)$ with the condition $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq 1, n > 3$ has only one positive integral solution given by $x_1 = n, x_2 = m_1,$ $x_3 = m_2, x_4 = 1 \dots, x_n = 1$

3. CONCLUSION : The Diophantine equation $a\prod x_i = b\sum x_i + k$ has been discussed for $k = 0, a = 1 = b$ and for other non-zero values of constants *a*, *b* and *k*. It has been shown that the Diophantine equations under consideration have only one solution each.

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STATISTICAL ASSESSMENT OF ADOPTION OF SELECTED AGRICULTURAL TECHNOLOGIES IN CHHATTISGARH STATE

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ABSTRACT :

The present study was conducted in different district of Chhattisgarh. The pattern of development of different districts was obtained with the help of composite index based on optimum combination of five agricultural indicators along with nine infrastructural indicators. The district-wise data for the year 2012-13 in respect of these indicators were utilized for 27 districts of the State. The pattern of development was estimated separately for agricultural sector and infrastructural facilities sector. The district of Raipur was ranked first in agricultural and infrastructural development. Wide disparities were observed in the pattern of development among different districts. Infrastructural facilities were found to be positively associated with the level of developments in agricultural sector and overall socio-economic field. Agricultural development was influencing the overall socio-economic development in the positive direction. The trends of composite and individual infrastructural indicators, fertilizer and HYVs have been examined and their impacts on agricultural productivity in Chhattisgarh state have been reported. The trend of composite infrastructure index has shown sharp fluctuations, while electrified villages and rural roads have indicated a rising trend.

Keywords : ***Developmental indicators, Composite index, Potential targets, Model districts.***

1. INTRODUCTION

Developmental programmes have been taken up in the country in a planned way through various Five Years Plans for enhancing the quality of life of people by providing basic necessities as well as effecting improvement in their social and economic well being. The green revolution in agricultural sector has enhanced the crop productivities and commendable progress in the industrial front has increased the quantum of manufactured goods but there is no indication that these achievements have been able to reduce substantially the level of regional disparities in terms of socio-economic development. Agricultural technologies include all kinds of improved techniques and practices which affect the growth of agricultural output. Due to data limitations, only five agricultural technologies have been used in this study for developing a new adoption index. These include (i) high-yielding varieties of seeds, (ii) chemical fertilizers, (iii) pesticides, (iv) plant protection implements, and (e) use of machinery, etc. By quality of improved input/output relationships, new technology tends to raise output and reduces average cost of production, which in turn results in substantial gains in farm income. Infrastructure plays a strategic role in producing large multiplier effects in the economy with growth in agriculture (Mellor, 1976). It is estimated that, across the world, 15 per cent of crop produce is lost between farm gate and consumer because of poor roads and inappropriate storage facilities. Binswanger *et al.* (1993), in a study of 13 Indian states, have found that investments in rural infrastructure lowered transportation costs, increased farmers' access to markets, and led to substantial agricultural expansion. Fan *et al.* (2000) have found that government expenditure on productivity growth was most effective when it was spent on rural infrastructure and agricultural research and development. Bhalla (2001) have also noted that the investment in irrigation and tube wells, and additional use of fertilizers and new seeds, helped in raising the productivity levels. They have also found higher production elasticity's for fertilizers, tube wells, tractors, irrigation and regulated markets. They have suggested that production was more responsive to modern inputs and infrastructure. The spread of technology in agriculture also depends on physical and institutional infrastructure.

According to Majumdar (2002), the transport infrastructure significantly affects the agricultural output and development in India.

For analyzing the impact of infrastructure on agricultural development, Thorat and Sirohi (2002) have used ten explanatory variables, viz. transport, power, irrigation, tractors, research, extension, access to agricultural credit societies, regulated and wholesale markets, access to fertilizer sale points and commercial banks, covering physical, financial and research infrastructures. They have reported that transport, power, irrigation and research were the four critical components affecting agricultural productivity significantly. With improved access to power, irrigation rises along with productivity. Development of transport facilitates access to fertilizer sale points, markets, credit facilities and extension services. The present study relates to Chhattisgarh state comprising of 27 districts. The nature of the study is pattern of agricultural development as well as infrastructural sectors.

2. MATERIALS AND METHODS

The adoption Indies of development were worked out for twenty seven districts separately for agricultural and industrial sector. For agricultural sector we had taken five indicators and infrastructural sector involves nine indicators. For this study, the districts were considered as the unit of analysis. Data collected from various publications, Government of Chhattisgarh were subjected to analyze through techniques given by Narain *et al* (1991). The developmental indicators considered for the study from the agricultural sectors and infrastructural sectors are as follows:

Table 2.1 Different parameters for adoption of agricultural technologies

Parameter	Identification
Percentage of gross cropped area under improved seeds	Seeds
Percentage of gross cropped area applied fertilizers	Fertilizers
Percentage of gross cropped area applied pesticides	Pesticides
Percentage of gross cropped area applied Plant protection implement	PPI
Percentage of gross cropped area tilled by tractors	Tractors

Source: DES , CG & Department of agriculture

Table 2.2 Infrastructural and developmental parameters considered for technology adoption

Parameter	Unit	Source	Identification
Percent rural literacy	No.	Census 2011	Literacy
Percent area connected with roads	No.	DES	Roads
Percent people below poverty line (Ration Card)	No.	CG food & civil supplies	BPL
Percent area connected with P & T	No.	Director General Post office	PT
Percent gross cropped area under irrigation	No.	Statistical Abstract 2012-13 DES	Irrigation
Agricultural and extension organizations (SAUs, ICAR, NGOs, public sector undertakings, state government, central universities and KVKs,) percent ten thousand hectares of net sown area (NSA)	No.	Government of Chhattisgarh and IGKV Krishi Darshika, 2012-13,	Organization
Credit per hectare of net sown area	Rs/Ha.	CG Apex Bank Coop Ltd.	Credit
Agricultural markets per thousand hectare of geographical area	No.	CG Cooperative bank ltd.	Markets
Electricity consumption per hectare of NSA	Th.Kwh.	Statistical Abstract 2012-13 DES	Electricity

2.1 Statistical analysis

2.1.1 The pattern of development

Let a set of n points represents districts 1,2,…,n for a group of k indicators 1,2,…,k. this can be represented by a matrix

$$[X_{ij}] = \begin{bmatrix} X_{11} & X_{12} \dots & X_{1j} \dots & X_{1k} \\ X_{21} & X_{22} \dots & X_{2j} \dots & X_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ X_{i1} & X_{i2} \dots & X_{ij} \dots & X_{ik} \\ \vdots & \vdots & \vdots & \vdots \\ X_{n1} & X_{n2} \dots & X_{nj} \dots & X_{nk} \end{bmatrix} nxk$$

Where:

i= 1,2,…,n

j= 1,2,…,k

As the development indicators included in the analysis were in different units measurement and since our objective was to arrive at a single composite index relating to the dimensions in question, there was a need for standardization of indicators. Hence, the standardization was done as shown below

$$Z_{ij} = \frac{X_{ij} - \overline{X_J}}{S_j}$$

Where (i= 1,2,…,n)

(j= 1,2,…,k)

$$\overline{X_J} = \sum_{i=1}^{n} \frac{X_{ij}}{n}$$

And

$$S_j^2 = \frac{1}{n} \sum_{i=1}^{n} (X_{ij} - \overline{X}_J)^2$$

The matrix (Z_{ij}) of standardization indicators is

$$[Z_{ij}] = \begin{bmatrix} Z_{11} & Z_{12} \dots & Z_{1j} \dots & Z_{1k} \\ Z_{21} & Z_{22} \dots & Z_{2j} \dots & Z_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ Z_{i1} & Z_{i2} \dots & Z_{ij} \dots & Z_{ik} \\ \vdots & \vdots & \vdots & \vdots \\ Z_{n1} & Z_{n2} \dots & Z_{nj} \dots & Z_{nk} \end{bmatrix} nxk$$

Where i= 1,2,…,n

j= 1,2,…,k

The best district for each indicator (with maximum/minimum standardized value depending upon the direction of the indicator) was identified and from this, the deviations of the value for each district were taken for all indicators in the following manner:

$$Ci = \left\{ \sum_{j=i}^{k} (Z_{ij} - Z_{oj})^2 \right\}^{\frac{1}{2}}$$

Where Z_{oj} denotes the standardized value of the jth indicator of the best district and C_i denotes the pattern of development of the ith district. The pattern of development was useful in identifying the district which serves as "model" and it also helps in fixing the potential target of each indicator for a given district.

2. 2.2 Estimation of composite index of development

The composite indexes of development were obtained through the following formula:

$$D_i = \frac{C_i}{C}$$

Where: $C = \bar{C} + 2s$

Here $\bar{C} = \sum_{i=1}^{n} \frac{C_i}{n}$ and $s = \left\{ \sum_{i=1}^{n} \frac{(C_i - \overline{C})^2}{n} \right\}^{1/2}$

The value of composite index is non-negative and it lies between 0 and 1. The value of composite index closer to one indicates the higher level of development while the value of index closer to 0 indicates the lower level of development.

2.2.3 The distance matrix

Using the standardized variables (Z_{ij}), the socio-economic distance between different districts may be obtained as follows:

$$D_{ip} = \{\sum_{j=i}^{k}(Z_{ij} - Z_{pj})^2\}^{1/2}$$

Where (i= 1,2,…,n) and (p= 1,2,…,n)

here $D_{ii} = 0 \; and \; D_{ip} = D_{pi}$

The distance matrix will take the from:

$$\begin{bmatrix} 0 & d_{12} \ldots & d_{13} \ldots & d_{1n} \\ d_{21} & 0 \ldots & Z_{2j} \ldots & Z_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ . & . \ldots & . \ldots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} \ldots & d_{nj} \ldots & 0 \end{bmatrix} nxn$$

The minimum distance (d_i, i= 1,2,…,n) for each row would be obtained from the distance matrix for computation of upper and lower limits (C.D.) as indicated below

$$C.D. = \bar{d} \pm 2\sigma_d$$

Where $\bar{d} = \sum_{i=1}^{n} \frac{d_i}{n}$ and $\sigma_d = \left\{\sum_{i=1}^{n} \frac{(d_i - \bar{d})^2}{n}\right\}^{1/2}$

2.2.4 Calculation of Classification

Table Simple ranking of district on the basis of composite indices would be sufficient for classificatory purposes. A suitable classification of the districts from the assumed distribution of the mean of the composite indices will provide more meaningful characterization of different stages of development. For relative comparison, it appears appropriate to assume that the districts having composite index more or equal to (Mean + SD) are developed and these districts are classified in category-I of the developed districts. Similarly the districts having composite indices between (Mean – SD) to (Mean) are classified as developing districts and put in category-II. In the same way, the districts having composite indices between (Mean) and (Mean - SD) are classified as poorly developed and put in category-III. An important aspect of the study is to find out the number of districts falling in different categories of development for each district.

Table 2.3 Calculation of Classification

Sector	Composite Indies		Limit of composite indies for different Category		
	Mean	SD	Category-I Developed	Category-II Developing	Category-III Poorly developed
Agriculture Development	0.28	0.24	Mean + SD	Category-I-Mean	Category-II-Mean
Infrastructural facility	0.47	0.18	Mean + SD	Category-I-Mean	Category-II-Mean

3. RESULTS AND DISCUSSIONS

3.1 Adoption Index and Agricultural Parameters

The composite indices of development have been worked out for different district separately for agricultural sector with five indicators and infrastructural sector along with nine indicators. Districts have been ranked on the basis of developmental indices. Higher value of composite index will indicate high level of development and smaller value of composite index will indicate low level of development. The composite indices are obtained by Narain *et al* (1991) method and Ratio Index Method and their comparison are given in Table 4.1. It may be seen from Table 4.1 that out of 27 districts, Raipur district was obtained first with maximum composite index and district Sukma was obtained ranked last in agricultural development sector. The mean values of first method the composite indices were obtained 0.28 varied from 0.02 to 0.93. In case of Ratio method, the mean value of composite indices was 0btained 0.017 varied from 0.01 to 0.04. In infrastructural sector composite indices are district Raipur was obtained first along with maximum composite index (0.96) followed by Durg (0.75) and district Sukma was obtained (0.13) ranked last in infrastructural sector. The mean values of first method the composite indices were obtained 0.47 varied from 0.13 to 0.96. In case of Ratio Index method, the mean value of composite indices was also obtained 0.10 varied from 0.04 to 0.18.

Table 3.1 The composite Indies of Development and Ranks of Major districts

Districts	Agricultural Development			Infrastructural facility			Productivity
	Rank	Narain C. I. Method	Ratio Index Method	Rank	Narain C. I. Method	Ratio Index Method	2012-13 Rs/ha
Koriya	21	0.08	0.01	19	0.42	0.05	1576
Surguja	18	0.12	0.02	21	0.38	0.11	2141
Balrampur	25	0.04	0.01	20	0.39	0.06	1779
Surajpur	16	0.14	0.01	23	0.31	0.09	1162
Jashpur	24	0.05	0.01	18	0.42	0.06	2200
Raigarh	13	0.26	0.02	14	0.49	0.09	2140
Korba	22	0.07	0.01	12	0.49	0.06	1593
Janjgir-Champa	10	0.38	0.02	5	0.60	0.11	4462
Bilaspur	5	0.47	0.02	6	0.58	0.16	4546
Mungeli	6	0.45	0.01	13	0.49	0.14	6362
Kabirdham	8	0.41	0.02	16	0.44	0.10	6596
Rajnandgaon	15	0.17	0.02	7	0.56	0.08	5610
Durg	4	0.49	0.02	2	0.75	0.15	10003
Bemetara	2	0.78	0.02	9	0.55	0.12	5880
Balod	11	0.35	0.03	3	0.63	0.11	10565
Raipur	1	0.93	0.03	1	0.96	0.18	8090
Baloda Bazar	3	0.60	0.02	10	0.50	0.12	5010
Gariyaband	7	0.43	0.01	11	0.50	0.11	5173
Mahasamund	12	0.32	0.01	8	0.55	0.11	4063
Dhamtari	9	0.40	0.04	4	0.62	0.12	8663
Kanker	19	0.09	0.02	15	0.46	0.09	2875
Bastar	14	0.17	0.02	24	0.30	0.13	3369
Kondagaon	23	0.06	0.02	22	0.32	0.11	2721
Narayanpur	26	0.03	0.01	25	0.17	0.06	857
Dantewada	20	0.09	0.01	17	0.42	0.05	132
Sukma	27	0.02	0.01	27	0.13	0.04	239
Bijapur	17	0.14	0.01	26	0.16	0.11	957

3.2 Different Stages of Development

On the basis of system of classification mentioned in section 2, the districts are put in 3 stages of development as developed district, developing district and poorly developed. In agriculture sector, out of 27 districts, 3 districts namely Bemetara, Raipur and Baloda Bazar are developed districts as compared to other districts. There are 10 districts, which are found to be developing and 14 districts are poorly developed for which special care is required while implementing the developmental programmes. As regards infrastructural facilities, 2 districts namely Durg and Raipur are developed. There are 12 districts in developing group and 13 districts in poorly developed group.

Necessary infrastructural facilities might be created in these districts for improvement in the level of development. It would be quite useful to study the district-wise information lying in different categories of development. This is important for micro level planning.

Table 3.2 Limits of Composite Indies for different stages of development

Sector	Composite Indies		Limit of composite indies for		
	Mean	SD	Developed	Developing	Poorly developed
Agriculture Development	0.28	0.24	0.52	0.28-0.51	0.01-0.27
Infrastructural facility	0.47	0.18	0.65	0.47-0.64	0.01-0.46

3.3 Inter-relationship among Different Sectors

Both the adoption indies were positively correlated (r = 0.83) and were significant. The Correlation between adoption index and individual agricultural parameters in respect of improved seed, fertilizer, pesticides, plant protection implementation and tractor was observed for different sets of districts. This was expected since grow and progress of agricultural development is very much influencing the positive direction and the correlation coefficient between the adoption Indies and agricultural parameters was found significant at 0.01 probability level. This indicates that these facilities are significantly affecting the level of development in agricultural sector. The best way to interpret the value of correlation ‘r’ is to square it to calculate r^2 called coefficient of determination, but scientists calls it "r squared". It is a value that ranges from zero to one, and is the fraction of the variance in the two variables that is “shared”. The r^2 of adoption index and improved seed was obtained 0.82 mean 82% of the variance in improved seed can be explained by variation in adoption. More simply, 82% of the variance is shared between improved seed and adoption index. The correlation coefficient between adoption index and individual agricultural parameters along with its test of significance and r^2 value are presented in table 3.3

Table 3.3 Correlation between adoption index and individual Infrastructural parameters for different sets of districts

Agricultural Indicators	**Correlation**	**Significance**	Infrastructural Parameter	**Correlation**	**Significance**
HYV	0.77**	4.57	Literacy	0.88**	9.37
Fertilizer	0.74**	4.14	Roads	0.74**	5.55
Pesticide	0.60**	2.24	BPL	0.55**	3.26
PPI	0.87**	7.47	Post & T	0.38*	2.05
Tractor	0.86**	7.31	Irrigation	0.82**	7.30
Irrigation	0.77**	4.57	Organization	0.63**	4.07
			Rs NSA	0.75**	5.67
			Market	0.82**	7.26
			Electricity	0.88**	9.37

Notes: r is the correlation coefficient and Sig refers to level of significance
*Significant at 0.05 level (2 tailed)
** Significant at 0.01 level (2 tailed)

3.4 Potential Target of Developmental Indicators for Poorly Developed District

It is quite useful and important to examine the extent of improvement required in various indicators for improving the level of development of poorly developed districts because it will help in bringing uniform development of district. Such information may help the planners and administrators to readjust the resources for reducing the disparities in the level of development among different districts. Special care should be taken in the developmental activities for poorly developed districts. The mean value of different indicators of model districts is taken as the potential target for the poorly developed district. Potential targets for some of the districts are quite high and improvements are needed in developmental programmes for achieving it. Action required for making improvement in the level of development of poorly developed district is given in table 4.4 and table 4.5 along with the present value.

Table 3.4 Potential targets and Present value of Development indicators of poorly developed districts for Agricultural sector

District	Improved Seed	Fertilizer	Pesticide	PPI	Tractor
Koriya	15.04	5.01	1.11	0.10	0.10
Surguja	19.80	6.60	2.64	0.32	0.20
Balrampur	1.70	1.61	2.03	0.16	0.12
Surajpur	28.00	2.49	3.01	0.08	0.17
Jashpur	3.98	1.33	0.78	0.17	0.15
Korba	0.85	0.28	7.67	0.81	0.11
Rajnandgaon	21.61	7.20	3.08	1.29	0.37
Kanker	5.36	1.79	5.54	1.09	0.19
Bastar	12.82	4.27	3.22	2.61*	0.29
Kondagaon	7.80	2.60	0.75	0.91	0.10
Narayanpur	0.66	0.22	0.19	0.62	0.02
Dantewada	1.29	0.43	1.08	1.72	0.04
Sukma	0.45	0.15	0.46	0.46	0.03
Bijapur	27.22	9.07	0.02	0.38	0.03
Potential Target	**45.96**	**20.53**	**18.65**	**2.18**	**0.61**

However, actual achievements of some of the poorly developed district were found to be better than potential targets in agricultural sector of the indicators eg. Bastar was having better level in PPI its potential targets. The district of Raigarh, Janjgir-Champa, Bilaspur, Mungeli, Kabirdham, Durg, Balod, Gariyaband and Dhamtari had achieved better level of improved seed then the corresponding targets. In infrastructural sector, some of the indicators eg. Surguja, Balrampur, Surajpur, Kabirdham, Sukma and Bijapur was having better level in BPL its potential targets. The district of Surguja, Jashpur, Bastar, Kodagaon, Dantewada had achieved better level of Post and Telegraph (P&T) then the corresponding targets. Sarguja, Bastar and Dantewada had performed better in the Agricultural Extension and Organization services then its target. The Kabirdham was having better level in agricultural credit its potential target. Balrampur, Kabirdham, Kanker and Kondagaon had performed better in the electricity services then its target.

Table 3.5 Potential targets and Present value of Development indicators of Low developed districts for Infrastructural Sector

District	Literacy	Roads	BPL	P/T	Irrigation	Organization	Credit	Market	Electricity
Koriya	70.64	17.54	16.84	10.72	8.00	1.60	1.58	3.04	2.03
Surguja	60.86	23.61	22.98*	27.45*	9.00	3.74*	2.14	5.17	16.99
Balrampur	60.95	10.82	25.60*	4.75	12.00	1.11	1.78	4.75	58.66*
Surajpur	57.98	14.65	23.69*	5.31	9.00	1.83	1.16	6.23	20.54
Jashpur	67.92	21.71	22.04	21.62*	3.00	1.86	0.22	3.05	16.69
Kabirdham	60.85	31.66	24.88*	7.04	30.00	0.95	6.60*	8.30	71.68*
Kanker	70.29	22.13	20.91	14.22	14.00	1.22	2.88	10.38	45.60*
Bastar	53.15	17.96	20.01	22.32*	3.00	4.77*	3.37	9.88	13.34
Kondagaon	56.21	33.29	20.29	17.61*	4.00	1.70	2.72	8.14	36.94*
Narayanpur	48.62	9.63	19.20	5.23	1.00	1.74	0.86	1.74	8.72
Dantewada	48.63	29.59	20.69	56.19*	0.00	3.98*	0.13	4.87	1.38
Sukma	34.81	8.19	23.06*	1.05	1.00	1.31	0.24	3.15	12.66
Bijapur	40.86	16.10	22.95*	9.65	5.00	1.23	0.96	2.28	11.18
Potential Target	**72.56**	**41.39**	**22.83**	**16.47**	**42.42**	**1.97**	**5.10**	**12.33**	**24.30**

4. CONCLUSIONS

The broad conclusions emerging from the study are as follows:

1. With respect to over all development of agricultural sector and infrastructural facilities the district of Bemetara, Raipur, Baloda Bazar and Durg district are found to be better developed as compared to other districts of the state.
2. Both the adoption indies were positively correlated (r = 0.83) and were significant. This was expected since grow and progress of agricultural development is very much influencing the positive direction and the correlation coefficient between the adoption Indies and agricultural parameters was found highly significant. This indicates that these facilities are significantly affecting the level of development in agricultural sector.
3. For enhancing the pattern of development of poorly developed district, model district have been identified and potential target of various developmental indicators have been obtained. The poorly developed district required various dimension in the developmental indicators.

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AVAILABILITY MODELING AND BEHAVIORAL ANALYSIS OF A SINGLE UNIT SYSTEM UNDER PREVENTIVE MAINTENANCE AND DEGRADATION AFTER COMPLETE FAILURE USING RPGT

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ABSTRACT :

In this paper Availability Modeling and Behavioral Analysis of a single unit system with Preventive Maintenance and Degradation after Complete Failure using Regenerative Point Graphical Technique (RPGT) is discussed. Initially the unit is working at full capacity which may have two type of failures, one is direct and second one is through partial failure mode. There is a single server (repairman) who inspects and repairs the unit as and when need arise. On partial failure (before complete failure) server repairs the unit which is perfect, but the unit and on complete failure the unit cannot be restored to its original capacity. If the server reports that unit is not repairable then it is replaced by a new one. Fuzzy concept is used to determine failure/working state of a unit. Taking failure rates exponential, repair rates general and taking into consideration various probabilities, a transition diagram of the system is developed to determine Primary circuits, Secondary circuits & Tertiary circuits and Base state. Problem is formulated and solved using RPGT. Repair and Failure are statistically independent. Expressions for systems parameters i.e. mean time to system failure, availability, and number of server visits and busy period of the server are evaluated to study the behavior of the system for steady state. Particular cases are taken to study the effect of failure and repair rates on mean time to system failure, availability, and expected number of server visits and busy period of the server. Profit optimization is also discussed. System behavior is discussed with the help of graphs and tables.

Keywords:- *Availability, Reliability, Primary Circuits, Tertiary Circuits, Degraded state, Base-State, Regenerative Point Graphical Technique (RPGT), MTSF, Busy period of the Server, Expected No. of Server's Visits, Fuzzy Logic, Steady State.*

INTRODUCTION: - Various systems are assembly of a number of units. A single unit system is widely used in a number of process industries and many more others. If a single unit fails then the whole system fails. Considering the importance of individual units in system Kumar, J. & Malik, S. C. [1] have discussed the concept of preventive maintenance for a single unit system. Liu., R. [2], Malik, S. C. [3], Nakagawa, T. and Osaki, S. [4] have discussed reliability analysis of a one unit system with un-repairable spare units and its applications. Goel, P. & Singh J. [5], Gupta, P., Singh, J. & Singh, I.P. [6], Kumar, S. & Goel, P. [8] and Gupta, V. K. [7] have discussed behavior with perfect and imperfect switch-over of systems using RPGT. Here we have discussed the behavioral analysis of a single unit system which have parallel sub units so it can work in reduced capacity as well as in full capacity. The system can fail from normal mode to complete failure directly or via partial failure. Most of the systems consist a number of units and one of these units may be important for working of system, hence need more care over other units. Further, there are units which may not be repaired to their original capacity. Since the capability of unit after repair depends on the repair mechanism adopted and unit may have increased failure rate on subsequent failures i.e. it is degraded after each repair. The system may go under imperfect repair on complete failure and unit is degraded but operative state is obtained again and again. Server inspects and repairs the unit as and when need arise. After a limiting situation, when no further repair is possible, then system is replaced by new one. In this paper Behavioral Analysis of a single unit system with Preventive Maintenance and Degradation after Complete Failure using Regenerative Point Graphical Technique (RPGT) is discussed. Initially the unit is working at full capacity which may have two type of failures, one is direct and second one is through partial failure mode. There is a single server (repairman) who inspects and repairs the unit as and when need arise. On partial failure (before complete failure) server repairs the unit which is perfect, but the unit and on complete failure the unit cannot be restored to its original capacity. If the server reports that unit is not repairable then it is replaced by a new one. Fuzzy concept is used to determine failure/working state of a unit. Taking failure rates exponential, repair rates general and taking into consideration various probabilities, a transition diagram of the system is developed to determine Primary circuits, Secondary circuits & Tertiary circuits and Base state. Problem is formulated and solved using RPGT. Repair and Failure are statistically independent. Expressions for systems parameters i.e. mean time to system failure, availability, and number of server visits and busy period of the server are evaluated to study the behavior of the system for steady state. Particular cases are taken to study the effect of failure and repair rates on mean time to system failure, availability, and expected number of server visits and busy period of the server. Profit optimization is also discussed. System behavior is discussed with the help of graphs and tables.

ASSUMPTIONS AND NOTATIONS: - The following assumptions and notations are taken: -

1. A single repair facility is available.
2. The distributions of failure times and repair times are exponential and general respectively and also different. Failures and repairs are statistically independent.
3. Repair is imperfect and repaired system is not good as new one on complete failure.
4. Nothing can fail when the system is in failed state.
5. The system is discussed for steady-state conditions.
6. Replacement of Un-repairable unit and repair facility is immediate.

$\overline{cycle}$: A circuit formed through un-failed states.

m-cycle : A circuit (may be formed through regenerative or non-regenerative / failed state) whose terminals are at the regenerative state m.

m- $\overline{cycle}$: A circuit (may be formed through un-failed regenerative or non-regenerative state) whose terminals are at the regenerative m state.

$\left(i^{sr} \to j\right)$: *r*-th directed simple path from *i*-state to j-state; r takes positive integral values for different paths from *i*-state to j-state.

$\left(\xi^{sff} \to i\right)$: A directed simple failure free path from ξ -state to i-state.

$V_{m,m}$: Probability factor of the state m reachable from the terminal state m of the m-cycle.

$V_{\overline{m,m}}$: Probability factor of the state m reachable from the terminal state m of the m-$\overline{cycle}$.

R_i (t) : Reliability of the system at time t, given that the system entered the un-failed regenerative state 'i' at t = 0.

A_i (t) : Probability of the system in up time at time 't', given that the system entered regenerative state 'i' at t = 0.

B_i (t) : Reliability that the server is busy for doing a particulars job at time 't'; given that the system entered regenerative state 'i' at t = 0.

V_i (t) : The expected no. of server visits for doing a job in (o,t] given that the system entered regenerative state 'i' at t = 0.

',' denote derivative

W_i (t) : Probability that the server is busy doing a particular job at time t without transiting to any other regenerative state 'i' through one or more non-regenerative states, given that the system entered the regenerative state 'i' at t = 0.

μ_i : Mean sojourn time spent in state i, before visiting any other states;

$$\mu_i = \int_0^\infty R_i(t)dt$$.

μ_i^1 : The total un-conditional time spent before transiting to any other regenerative states, given that the system entered regenerative state 'i' at t=0.

n_i : Expected waiting time spent while doing a given job, given that the system entered regenerative state 'i' at t=0; $\eta_i = W_i^*(0)$.

ξ : Base state of the system.

f_j : Fuzziness measure of the j-state.

λ / λ_i : Constant failure rate of units A/ degraded unit A for $1 \le i \le 3$.

A/$\bar{A}$/a : Unit in full capacity working / reduced state / failed state.

Taking into consideration the above assumptions and notations the Transition Diagram of the system is given in Figure 1.

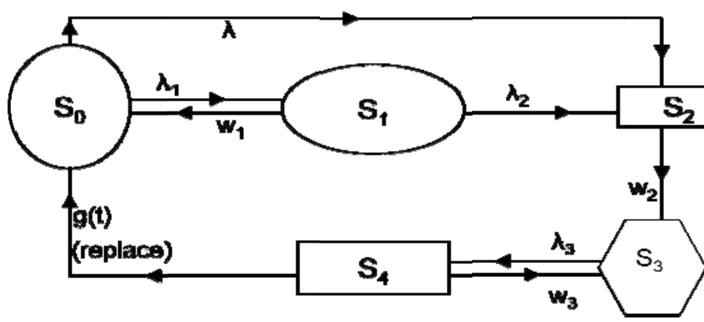

Figure-1

State	Symbol
Regenerative State/Point	⊙
Up-state	○
Failed State	▭
Reduced State/ Degraded State	⬡

Table 1

The system can be in any of the following states with respect to the above symbols.

S_0 = A S_1 = $\bar{A}$ S_2 = a

S_3 = $\bar{A}_1$ S_4 = a_1

The failure rate of unit A to $\bar{A}$ is λ_1 and from $\bar{A}$ to 'a' is λ_2

1. Failure rate of unit '$\bar{A}_1$' to 'a_1' is λ_3.
2. Failure rate of unit 'A' to 'a' is λ.
3. Repair rates of unit $\bar{A}$ to A is w_1
4. Repair rate of unit a to $\bar{A}_1$ is w_2
5. Repair rate of unit a_1 to $\bar{A}_1$ is w_3
6. g(t) : Probability density function that Un-repairable unit A is replaced by new one.

States S_0, S_1, S_3 and S_4 are regenerative states.

1.3 Transition Probabilities and the Mean sojourn times.

$q_{i,j}(t)$ **:** Probability density function (p.d.f.) of the first passage time from a regenerative state 'i' to a regenerative state 'j' or to a failed state 'j' without visiting any other regenerative state in (0,t].

$p_{i,j}$ **:** Steady state transition probability from a regenerative state 'i' to a regenerative state 'j' without visiting any other regenerative state. $p_{i,j}$ = $q^*_{i,j}(0)$; where * denotes Laplace transformation.

$q_{i,j}^{(t)}$	$P_{ij} = q^*_{i,j}{}^{(t)}$
$q_{0,1}=\lambda_1 e^{-(\lambda_1+\lambda)t}$ $q_{0,2}=\lambda e^{-(\lambda_1+\lambda)t}$	$p_{0,1}=\lambda_1/(\lambda+\lambda_1)$ $p_{0,2}=\lambda/(\lambda+\lambda_1)$
$q_{1,0}=w_1 e^{-(w_1+\lambda_2)t}$ $q_{1,2}=\lambda_2 e^{-(w_1+\lambda_2)t}$	$p_{1,0}=w_1/(w_1+\lambda_2)$ $p_{1,2}=\lambda_2/(w_1+\lambda_2)$
$q_{2,3}=w_2 e^{-w_2 t}$	$p_{2,3}=w_2/w_2=1$
$q_{3,4}=\lambda_3 e^{-\lambda_3 t}$	$p_{3,4}=\lambda_3/\lambda_3=1$
$q_{4,3}=w_3 e^{-w_3 t}$	$p_{4,3}=w_3/w_3=1$

Table 2

Here we see that $p_{0,1}+p_{0,2}=1$, $p_{1,0}+p_{1,2}=1$,

Hence it is verified that for each state total state probability is 1.

Mean Sojourn Times

$R_i(t)$: Reliability of the system at time t, given that the system in regenerative state i.

μ_i : Mean sojourn time spent in state i, before visiting any other states;

$R_i(t)$	$\mu_i = R_i^*(0)$
$R_0^{(t)} = e^{-(\lambda_1+\lambda)t}$	$\mu_0 = 1/(\lambda+\lambda_1)$
$R_1^{(t)} = e^{-(w_1+\lambda_2)t}$	$\mu_1 = 1/(w_1+\lambda_2)$
$R_2^{(t)} = e^{-w_2 t}$	$R_2(0) = \mu_2 = 1/w_2$
$R_3^{(t)} = e^{-\lambda_3 t}$	$R_3(0) = \mu_3 = 1/\lambda_3$
$R_4^{(t)} = e^{-w_3 t}$	$R_4^{(0)} = \mu_4 = 1/w_3$

Table 3

The path probabilities for the steady state are given w.r.t. Base State '0' are

$V_{0,0} = (0,1,0)+(0,1,2,3,4,0)+(0,2,3,4,0) = p_{0,1}p_{1,0} + p_{0,1}\, p_{1,2}\, p_{2,3}\, p_{3,4}\, p_{4,0} + p_{0,2}\, p_{2,3}\, p_{3,4}\, p_{4,0}$

$= \lambda_1/(\lambda+\lambda_1).w_1/(w_1+\lambda_2)+\lambda_1/(\lambda+\lambda_1).\lambda_2/(w_1+\lambda_2)+ \lambda/(\lambda+\lambda_1) = 1$

$V_{0,1} = p_{0,1} = \lambda_1/(\lambda+\lambda_1)$

$V_{0,2} = (0,1,2) + (0,2) = p_{0,1}\, p_{1,0} + p_{0,2} = \lambda_1/(\lambda+\lambda_1)\ \lambda_2/(w_1+\lambda_2)+ \lambda/(\lambda+\lambda_1)$

$V_{0,3} = (0,1,2,3) + (0,2,3) = p_{0,1}\, p_{1,2}\, p_{2,3} + p_{0,2}\, p_{2,3} = \lambda_1/(\lambda+\lambda_1)\ \lambda_2/(w_1+\lambda_2)\ (1+\lambda)/(\lambda+\lambda_1)$

$V_{0,4} = (0,1,2,3,4) + (0,2,3,4) = p_{0,1}\, p_{1,2}\, p_{2,3}\, p_{3,4} + p_{0,2}\, p_{2,3}\, p_{3,4} = \lambda_1/(\lambda+\lambda)\ \lambda_2/(w_1+\lambda_2) + \lambda/(\lambda+\lambda_1)$

(i). Mean Time to System Failure (MTSF) (T_0): The regenerative un-failed states to which the system can transit(initial state '0'), before entering any failed state are: 'i' = 0,1,2,3,4 taking 'ξ' = '0'.

$$\textbf{MTSF }(T_0) = \left[\sum_{i,sr}\left\{\frac{\left\{pr\left(\xi \xrightarrow{sr(sff)} i\right)\right\}\mu i}{\prod_{m_1\neq\xi}\left\{1-V_{\overline{m_1 m_1}}\right\}}\right\}\right] \div \left[1-\sum_{sr}\left\{\frac{\left\{pr\left(\xi \xrightarrow{sr(sff)} \xi\right)\right\}}{\prod_{m_2\neq\xi}\left\{1-V_{\overline{m_2 m_2}}\right\}}\right\}\right]$$

$= [(0,0)\mu_0 + (0,1)\mu_1]/\ [1-(0,1,0) = [1/(\lambda+ \lambda_1)+ \lambda_1/(\lambda+ \lambda_1)\ 1/(w_1+\lambda_2)]/$

$[1-\lambda_1(\lambda+\lambda_1)\ w_1/(w_1+\lambda_2)]$

$= (w_1+\lambda_2+\lambda_1)/[(\lambda+\lambda_1)(w_2+\lambda_2)- \lambda_1 w_1 = (\lambda_1+\lambda_2+w_1)/(\lambda w_1 + \lambda\lambda_2+ \lambda_1\lambda_2)$

(ii). Availability of the System: The regenerative states at which the system is available are 'j' = 0,1,3 and the regenerative states are 'i' = 0 to 4 taking 'ξ' = '0' the total fraction of time for which the system is available is given by

$$A_0 = \left[\sum_{j,sr}\left\{\frac{\{pr(\xi^{sr\rightarrow} j)\} f j, \mu j}{\prod_{m_1\neq\xi}\left\{1-V_{\overline{m_1 m_1}}\right\}}\right\}\right] \div \left[\sum_{i,s_r}\left\{\frac{\{pr(\xi^{sr\rightarrow} i)\}\mu_i^1}{\prod_{m_2\neq\xi}\left\{1-V_{\overline{m_2 m_2}}\right\}}\right\}\right]$$

$$A_0 = \left[\sum_j V_{\xi,j}\,,\, f_j, \mu_j\right] \div \left[\sum_i V_{\xi,i}\,,\, f_j, \mu_i^1\right]$$

$= V_{0,0}\, f_0\, \mu_0+V_{0,1}f_1\mu_1+ V_{0,3}\, f_3\, \mu_3\,]/(V_{0,0}\mu_0^1+ V_{0,1}\mu_1^1 + V_{0,2}\mu_2^1 + V_{0,3}\mu_3^1 + V_{0,4}\mu_4^1)$

Taking $f_0 = f_1 = f_3 = 1$, $\mu_i^1 = \mu_i$

$= (V_{0,0}\mu_0 + V_{0,1}\mu_1+ V_{0,3}\mu_3)(V_{0,0}\mu_0 + V_{0,1}\mu_1 + V_{0,2}\mu_2 + V_{0,3}\mu_3 + V_{0,4}\mu_4)$

$= [1/(\lambda+\lambda_1)+[1/(\lambda+\lambda_1)1/(w_1+\lambda_2)]+[\lambda_1/(\lambda+\lambda_1)\lambda_2/(w_1+\lambda_2)+\lambda/(\lambda+\lambda_1)1/\lambda_3]/$

$[\ 1/(\lambda+\lambda_1)+ \lambda_1/(\lambda+\lambda_1)\ 1/(w_1+\lambda_2)]$

$= \lambda_1/(\lambda+\lambda_1)\ \lambda_2/(w_1+\lambda_2)+ \lambda/(\lambda+\lambda_1)1/w_3+1/w_2+1/\lambda_3$

(iii). Busy Period of the Server: The regenerative states where server 'j' = 1,2,3,4 taking base state ξ = '0', the total fraction of time for which the server remains busy is

$$B_0=\left[\sum_{j,sr}\left\{\frac{\{pr(\xi^{sr\to}j)\},nj}{\Pi_{m_1\neq\xi}\{1-V_{\overline{m_1m_1}}\}}\right\}\right]\div\left[\sum_{i,s_r}\left\{\frac{\{pr(\xi^{sr\to}i)\}\mu_i^1}{\Pi_{m_2\neq\xi}\{1-V_{\overline{m_2m_2}}\}}\right\}\right]$$

$$B_0 = \left[\sum_j V_{\xi,j}\,, n_j\right] \div \left[\sum_i V_{\xi,i}\,, \mu_i^1\right]$$

$= V_{0,1}n_1+V_{0,2}n_2+V_{0,4}n_4/V_{0,0}\,\mu^1{}_0+V_{0,1}\mu^1{}_1+V_{0,2}\mu^1{}_2+V_{0,3}\mu^1{}_3+V_{0,4}\,\mu^1{}_4$

$= V_{0,1}\mu_1+V_{0,2}\mu_2+V_{0,4}\mu_4/V_{0,0}\,\mu_0+V_{0,1}\mu_1+V_{0,2}\mu_2+V_{0,3}\mu_3+V_{0,4}\,\mu_4$

$= [\lambda_1/(\lambda+\lambda_1)1/(w_1+\lambda_2)]+[\lambda_1/(\lambda+\lambda_1)\lambda_2/(w_1+\lambda_2)+\lambda/(\lambda+\lambda_1)](1/w_2+1/w_3)/$

$[1/(\lambda+\lambda_1)+ \lambda_1/[(\lambda+\lambda_1)\; 1/(w_1+\lambda_2)+ \lambda_1/(\lambda_1+\lambda)\lambda_2/(w_1+\lambda_2)+\lambda/(\lambda+\lambda_1)(1/w_2+1/\lambda_3+1/w_3)]$

(iv). Expected Number of Server's Visits (V_0): The regenerative states where the repair man do this job j = 1 Taking base state 'ξ' = '0', the number of visit by the repair man is given by

$$V_0=\left[\sum_{j,sr}\left\{\frac{\{pr(\xi^{sr\to}j)\}}{\Pi_{k_1\neq\xi}\{1-V_{\overline{k_1k_1}}\}}\right\}\right]\div\left[\sum_{i,s_r}\left\{\frac{\{pr(\xi^{sr\to}i)\}\mu_i^1}{\Pi_{k_2\neq\xi}\{1-V_{\overline{k_2k_2}}\}}\right\}\right]$$

$$V_0 = \left[\sum_j V_{\xi,j}\right] \div \left[\sum_i V_{\xi,i}\,, \mu_i^1\right]$$

$= (V_{0,1} + V_{0,2} + V_{0,4})/(V_{0,0}\,\mu^1{}_0 + V_{0,1}\,\mu^1{}_1 + V_{0,2}\,\mu^1{}_2 + V_{0,3}\,\mu^1{}_3 + V_{0,4}\,\mu^1{}_4)$

(where $\mu_j{}^1 = \mu_j$ for all j)

$= (V_{0,1} + V_{0,2} + V_{0,3}) / (V_{0,0}\,\mu_0 + V_{0,1}\,\mu_1 + V_{0,2}\,\mu_2 + V_{0,3}\,\mu_3 + V_{0,4}\,\mu_4)$

$= [\lambda_1/(\lambda+\lambda_1)+[\lambda_1/(\lambda+\lambda_1)\lambda_2/(w_1+\lambda_2)+ \lambda/(\lambda+\lambda_1)]+[\lambda_1/(\lambda+\lambda_1)\,\lambda_2/(w_1+\lambda_2)+\lambda/(\lambda+\lambda_1)]/$

$[1/(\lambda+\lambda_1)+\lambda_1/(\lambda+\lambda_1)\; 1/(w_1+\lambda_2)+ \lambda_1/(\lambda+\lambda_1)\;\lambda_2/(w_1+\lambda_2)+ \lambda/(\lambda+\lambda_1)(1/w_2+1/\lambda_3+1/w_3)]$

Particular Cases

Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $w_1 = w_2 = w_3 = w$,

MTSF (T_0) = $(\lambda+\lambda+w)/(\lambda w+\lambda^2+\lambda^2) = (2\lambda+w)/\lambda(2w+\lambda) = 1/\lambda$

$dT_0/d\lambda = -1/\lambda^2$ i.e. MTSF inversely varies with failure rate λ.

If λ = 0, then MTSF is infinity

Availability (A_0) = $[(1/2\lambda+1/2(w+\lambda)+(1/2\,\lambda/(w+\lambda)+1/2)1/\lambda)]\div$

$[(1/2\lambda)+1/2\;1/(w+\lambda)+(1/2\,\lambda/(w+\lambda)+1/2)\,(1/w+1/w+1/\lambda)]$

$= [1/\lambda+1/(w+\lambda)+(2\lambda+w)/(w+\lambda)1/\lambda]\div$

$[1/\lambda+1/(w+\lambda)+(2\lambda+w)/(w+\lambda)(2\lambda+w)/(w+\lambda)]$

$= [(w+\lambda+\lambda+2\lambda+w)/(w+\lambda)\lambda]/[(w+\lambda)w+w\lambda+(2\lambda+w)^2/(w+\lambda)w\lambda]$

$= (2w^2+4\lambda w)/(2w^2+6w\lambda+4\lambda^2) = (w^2+2w\lambda)/(w^2+3w\lambda+2\lambda^2)$

Availability of system for different rate of failures and different repair rate given by table.

Availability Table

A	ω = 0.60	ω = 0.70	ω = 0.75	ω = 0.80	ω = 0.90
λ = 0.06	0.9011	0.9210	0.9259	0.9302	0.9375
λ = 0.07	0.8955	0.9090	0.9146	0.9195	0.2278
λ = 0.08	0.8823	0.8974	0.9036	0.9090	0.9183
λ = 0.09	0.8728	0.8860	0.8928	0.8989	0.9091
λ = 0.10	0.8571	0.8750	0.8823	0.8889	0.9082

Table 4

Availability Graph

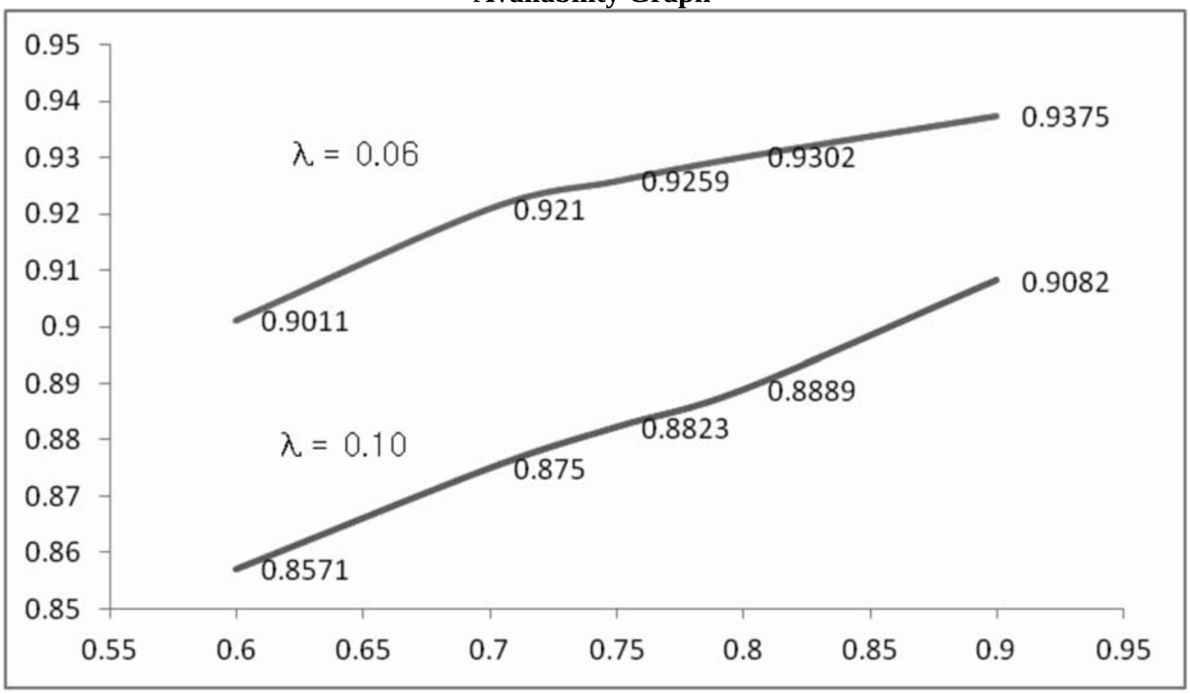

Figure 2

Busy Period of Server (B_0)

= [(1/2 1(w+λ)+(1/2 λ/(w+λ)+1/2)(2/w)]÷[(1/2λ+1/2(w+λ)+(λ/2(w+λ)+1/2)(2/w+1/λ)]

= [1/(w+λ)+(2λ+w)/(λ+w)2/w]÷ [1/λ+1/(w+λ)+(2λ+w)/(w+λ)(2/w+1/λ)]

Busy period for different rate of failure and repair rates are given below

Busy Period Table

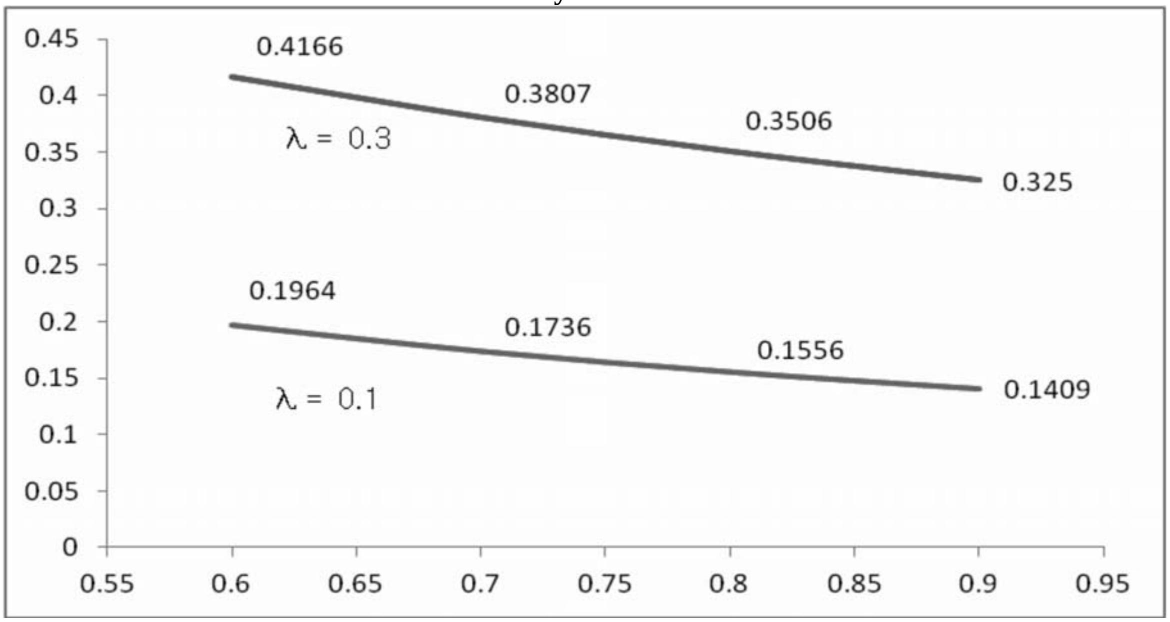

figure 3

Expected Number of Server Visits (V_0)

$= [1/2+(1/2\ \lambda/(w+\lambda)+1/2)2]\div[(1/2\lambda+1/2\ 1/(\lambda+w)+1/2)(2/w)+1/\lambda)]$

$= [(1+2\lambda+w)/(w+\lambda)^2]/[(1/\lambda+1/(\lambda+w)+(2\lambda+w)/(\lambda+w)$

$(2\lambda+w)/(\lambda+w)[(5\lambda+3w)/(w+\lambda)]/[w(\lambda+w)+w\lambda+(2\lambda+w)^2]$

$= [(\lambda+w)(\lambda w)][\lambda w(5\lambda+3w)]/(2\lambda^2+4\lambda^2+6w\lambda)$

Expected Number of Visits (V_0) for different failure rates and different rates are given below

Expected Number of Visits Table

V_0	ω = 0.6	ω = 0.7	ω = 0.8	ω = 0.9
λ = 0.1	0.1312	0.1289	0.1263	0.1232
λ = 0.2	0.2817	0.2643	0.2261	0.2100
λ = 0.3	0.3761	0.3215	0.3027	0.2852

Table 5

The expected number of visits for the period of observation is obtained by multiplying the tabular value with the time of observation and rounding off to whole number.

Expected Number of Visits Graph

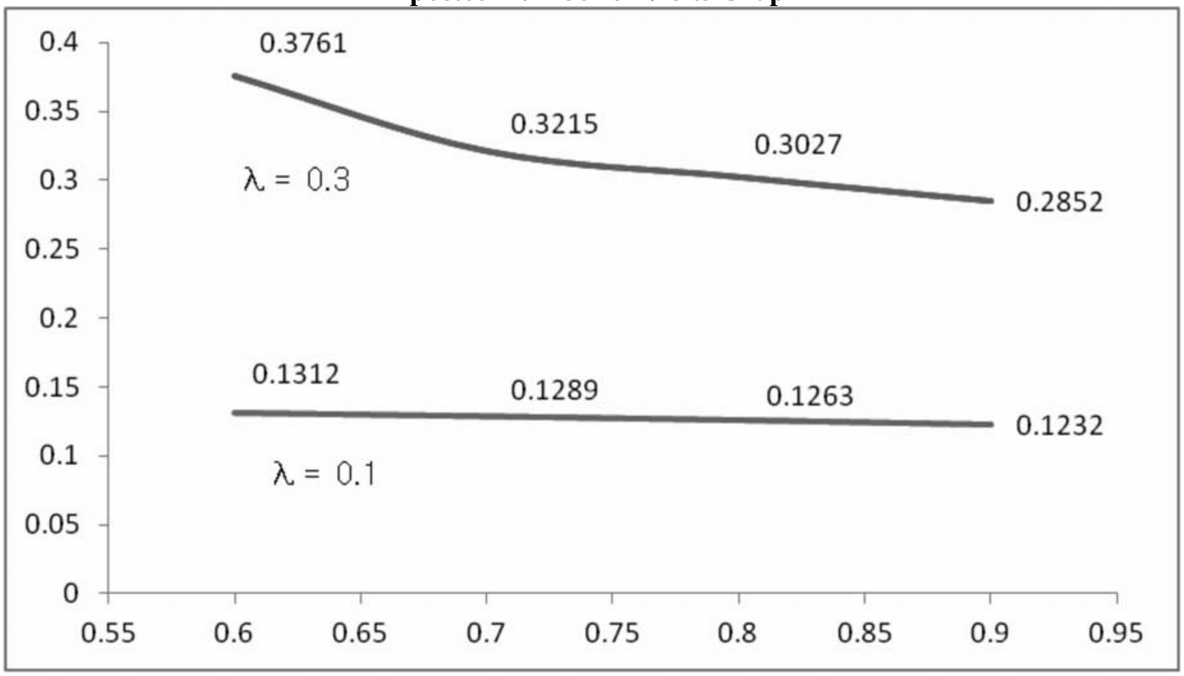

Figure 4

PROFIT FUNCTION OF THE SYSTEM

Profit analysis of the system can be done by using the profit function

$$P_0 = C_1A_0 - C_2B_0 - C_3V_0$$

Where C_1 = Revenue per unit time the system is available

C_2 = Cost per unit time the server remain busy for the repair.

C_3 = Cost of visit of the server.

CONCLUSION: - From the tables and analytical discussions, we see that increase in the repair rate (w) increases the availability of the system and the mean time to system failure whereas increase in failure rate decrease the availability and mean time to system which should be so practically. The Regenerative Point Graphical Technique is useful to evaluate the key parameters of the system in a simple way, without writing any state equations and without doing any lengthy and cumbersome calculations. We also see from the busy period of server table that when we increase the repair rate busy period of server decrease and when we increase the failure rate then busy period of server also increase. Also from expected number of inspections by the repair man table we see that when we increase the repair rate the expected number of inspections by repairman increase and when we increase the failure rate then the expected number of inspections by repairman is also increase. In future, Researchers can evaluate the parameters, when repair rate sand failure rate are variable and also discuss the cost and profit benefit analysis. Further results can also be applied to find the waiting time of units and number of server visits, as if the states where the server is on prime visit or on a secondary visit are determined separately using the formula. Since the cost of secondary visit is usually less than primary visit of server, therefore the system can be run with low maintenance cost. Various system parameters can also be evaluated taking any state as base state. As failure rates are beyond control, so determine the repair rates and reduce the fixing the target of availability management can cost of maintenance.

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THERMAL INSTABILITY IN ANISOTROPIC POROUS MEDIUM LAYER SATURATED BY A NANO-FLUID : BRINKMAN MODEL

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ABSTRACT :
In this paper, the onset of convection in horizontal layer of porous medium saturated by a nanofluid has been studied analytically incorporating the effects of Brownian motion and thermophoresis. Here the study is extended to the Brinkman model by introducing an additional parameter Darcy number.
Keywords: ***Nanoparticles, Nanofluid, Anisotropic Porous medium and Darcy number.***
Keywords : ***Developmental indicators, Composite index, Potential targets, Model districts.***

1. INTRODUCTION:

The term “nanofluid” refers to a liquid containing a suspension of sub-micronic solid particles (nanoparticles). The term was coined by **Choi [3]**. The characteristic feature of nanofluids in thermal conductivity enhancement, phenomenon observed by **Masuda et al. [5]**. This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems **(Buongiorno and Hu [2])**.

A comprehensive survey of convective transport in nanofluids was made by **Buongiorno [1]** who says that a satisfactory explanation for the abnormal increase of the thermal conductivity and viscosity is yet to be found. He focused on the further heat transfer enhancement observed in convective situations. Buongiorno notes that several authors have suggested that convective heat transfer enhancement could be due to the dispersion of the suspended nanoparticles, but he argues that this effect is too small to explain the observed enhancement. Buongiorno also concludes that turbulence is not affected by the presence of the nanoparticles; therefore, it cannot explain the observed enhancement particle rotation has also been proposed as a cause of heat transfer enhancement, but Buongiorno calculates that this effect is too small to explain the effect. With dispersion, turbulence and particle rotation ruled out as significant agencies for heat transfer enhancement, Boungiorno proposed a new model based on the mechanics of the nanoparticle/ base-fluid relative velocity. **Buongiorno [1]** noted that the nanoparticle absolute velocity could be viewed as the sum of the base fluid velocity and a relative velocity (that he calls the slip velocity). He considered, in turn, seven slip mechanisms: inertia, Brownian diffusion, thermophoresis, diffusionphoresis, magnus effect, fluid drainage and gravity settling. After examining each of these in turn, he concluded that in the absence of turbulent effects, it is the Brownian diffusion and the thermophoresis that will be important. Buongiorno proceeded to write down conservation equations based on these two effects.

The Benard problem (the onset of convection in a horizontal layer uniformly heated from below) for a nanofluid was studied by **Tzou [10,11]** and **Nield and Kuznetsov [7]** on the basis of the transport equations of **Buongiorno [1]**. The corresponding problem for flow in a porous medium (the Horton-Rogers-Lapwood problem) was studied by **Nield and Kuznetsov [8]** using the Darcy model. Harikishan etal [12] explored the thermal instability of stratified discuss shear flow and range a instable mode for weak thermometric conductivity In this paper, the study is extended to the Brinkman model. This necessitates the introduction of an additional parameter, namely, a Darcy number.

List of Symbols:

D_B Brownian diffusion coefficient

D_T Thermophoretic diffusion coefficient

Da Darcy number

g Gravitational acceleration

g Gravitational acceleration vector

H Dimensional layer depth

k_m Effective thermal conductivity of the porous medium

(k_x, k_y, k_z) Anisotropic effect

L_e Lewis number

N_A Modified diffusivity ratio

N_B Modified particle-density increment

p Dimensionless pressure, $p^* K / \mu\alpha_m$

p_r Prandtl number

Ra Thermal Rayleigh-Darcy number

Rm Basic density Rayleigh number

Rn Concentration Rayleigh number

2. FORMULATION OF THE PROBLEM:

It is assumed that nanoparticles are suspended in the nanofluid using either surfactant or surface charge technology. This prevents the particles from agglomeration and deposition on the porous matrix. We select a coordinate frame in which the z-axis is aligned vertically upward. We consider a horizontal layer of anisotropic porous medium confined between the planes $z = 0$ and $z = H$. Asterisks are used to denote the dimensional variables. Each boundary wall is assumed to be impreamble and perfectly thermally conducting. The temperatures at the lower and upper wall are taken to be T_h and T_c, the former being the greater. The Oberbeck-Boussinesq approximation is employed. The reference temperature is taken to be T_c. In the linear theory being applied here, the temperature change in the fluid is assumed to be small in comparison with T_c.

Homogeneity and local thermal equilibrium in the anisotropic porous medium are assumed. We are aware that thermal lagging between the particles and the fluid has been proposed as an explanation of the increased thermal conductivity that has been observed in nanofluids, but this is not our concern here. The extra complication of local thermal non-equilibrium could well be the subject of future research.

The Darcy velocity is denoted by v_D. The following four field equations embody the conservation of total mass, momentum, thermal energy and nanoparticles, respectively. The field variables are the Darcy velocity v_D, the temperature T and the nanoparticle volume fraction ϕ.

$$\nabla . v = 0 , \qquad \dots (1)$$

$$\frac{\rho f}{\varepsilon}\frac{\partial V}{\partial t} = -\nabla p + \mu \nabla^2 v' - \frac{\mu v}{(k_x, k_y, k_z)} + \left[\phi\rho_p + (1-\phi)\rho_f \left\{1-\beta(T-T_C)\right\}\right] g , \qquad \dots (2)$$

$$\left(\rho c\right)_m \frac{\partial T}{\partial t} + \left(\rho c\right)_f \, v.\nabla T = k_m \nabla^2 T + \varepsilon\left(\rho c\right)_p \left[D_B \nabla\phi.\nabla T + \left(\frac{D_T}{T_C}\right)\nabla T.\nabla T \right], \quad \dots (3)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon} v.\nabla\phi = D_B \nabla^2\phi + \left(\frac{D_T}{T_C}\right)\nabla^2 T, \quad \dots (4)$$

$$v = \left(u, v, w\right).$$

Here ρ_f, μ and β are the density, viscosity and volumetric volume expansion coefficient of the fluid, while ρ_p is the density of the particles. The gravitational acceleration is denoted by **g**. We have introduced the effective viscosity $\tilde{\mu}$, the effective heat capacity $\left(\rho c\right)_m$ and the effective thermal conductivity k_m of the anisotropic porous medium. The coefficient that appear in equations (3) and (4) are the Brownian diffusion coefficient D_B and the thermophoretic diffusion coefficient D_T. Details of the derivation of equations (3) and (4) are given in the articles by **Buongiorno [1]; Tzou [10,11], Nield and Kuznetsov [7,8]**. The flow is assumed to be slow so that an advective term and a Forchheimer quadratic drag term do not appear in the momentum equation.

We assume that the temperature and the volumetric fraction of the nanoparticles are constant on the boundaries. Thus, the boundary conditions are

$$w = 0,\ \frac{\partial w}{\partial z} + \lambda_1 H \frac{\partial^2 w}{\partial z^2} = 0,\ T = T_h,\ \phi = \phi_0 \text{ at } z = 0, \quad \dots (5)$$

$$w = 0,\ \frac{\partial w}{\partial z} - \lambda_2 H \frac{\partial^2 w}{\partial z^2} = 0,\ T = T_C,\ \phi = \phi_1 \text{ at } z = H. \quad \dots (6)$$

The parameters λ_1 and λ_2 each take the value 0 for the case of a rigid boundary and ∞ for a free boundary. We introduce dimensionless variables as follows. We define

$$\left(x^*, y^*, z^*\right) = \frac{\left(x, y, z\right)}{H},\ t^* = \frac{t\alpha_m}{\sigma H^2},\ \left(u^*, v^*, w^*\right) = \frac{\left(u, v, w\right)H}{\alpha_m},$$

$$p^* = \frac{pk}{\mu\alpha_m},\ \phi^* = \frac{\phi - \phi_0}{\phi_1 - \phi_0},\ T^* = \frac{T - T_C}{T_h - T_C}, \quad \dots (7)$$

where $\alpha_m = \dfrac{k_m}{\left(\rho c\right)_f}$, $\sigma = \dfrac{\left(\sigma c\right)_m}{\left(\rho c\right)_f}$. … (8)

Equation (1) to equation (6) take the form :

$$\nabla.V = 0 \quad \dots (9)$$

$$\left.\begin{aligned}
\frac{Da_x}{P_r}\frac{\partial u}{\partial t} &= -\nabla p + Da_x \nabla^2 u - u - Rm_x\hat{\rho}_z + Ra_x\hat{\rho}_z + Rn_x\phi\hat{\rho}_z, \\
\frac{Da_y}{P_r}\frac{\partial v}{\partial t} &= -\nabla p + Da_y \nabla^2 v - v - Rm_y\hat{\rho}_z + Ra_y\hat{\rho}_z + Rn_y\phi\hat{\rho}_z, \\
\frac{Da_z}{P_r}\frac{\partial v}{\partial t} &= -\nabla p + Da_z \nabla^2 w - w - Rm_z\hat{\rho}_z + Ra_z\hat{\rho}_z + Rn_z\phi\hat{\rho}_z,
\end{aligned}\right\} \quad \dots (10)$$

$$\frac{\partial T}{\partial t} + V.\nabla T = \nabla^2 T + \frac{N_B}{L_e}\nabla\phi.\nabla T + \frac{N_A N_B}{L_e}\nabla T.\nabla T, \quad \dots (11)$$

$$\frac{1}{\sigma}\frac{\partial\phi}{\partial t} + \frac{1}{\varepsilon}V.\nabla\phi = \frac{1}{L_e}\nabla^2\phi + \frac{N_A}{L_e}\nabla^2 T, \quad \dots (12)$$

$$w=0,\ \frac{\partial w}{\partial z}+\lambda_1\frac{\partial^2 w}{\partial z^2}=0,\ T=1,\ \phi=0\ \text{at}\ z=0, \quad \ldots(13)$$

$$w=0,\ \frac{\partial w}{\partial z}-\lambda_2\frac{\partial^2 w}{\partial z^2}=0,\ T=0,\ \phi=1\ \text{at}\ z=1. \quad \ldots(14)$$

Here

$$p_r=\frac{\mu}{\rho_f\alpha_m}, Da_x=\frac{\tilde{\mu}k_x}{\mu H^2}, L_e=\frac{\alpha_m}{D_B}, Da_y=\frac{\tilde{\mu}k_y}{\mu H^2}, Da_z=\frac{\tilde{\mu}k_z}{\mu H^2} \quad \ldots(15)$$

$$\left.\begin{aligned} Ra_x&=\frac{\rho g\beta k_x H(T_h-T_C)}{\mu\alpha_m},\\ Ra_y&=\frac{\rho g\beta k_y H(T_h-T_C)}{\mu\alpha_m},\\ Ra_z&=-\frac{\rho g\beta k_z H(T_h-T_C)}{\mu\alpha_m},\end{aligned}\right\} \quad \ldots(16)$$

$$\left.\begin{aligned} Rm_x&=\frac{\{\rho g\beta k_x H(T_h-T_C)\}gk_xH}{\mu\alpha_m},\\ Rm_y&=\frac{\{\rho g\beta k_y H(T_h-T_C)\}gk_yH}{\mu\alpha_m},\\ Rm_z&=-\frac{\{\rho g\beta k_z H(T_h-T_C)\}gk_zH}{\mu\alpha_m},\end{aligned}\right\} \quad \ldots(17)$$

$$\left.\begin{aligned} Rn_x&=\frac{(\rho_p-\rho)(\phi_1-\phi_0)gk_xH}{\mu\alpha_m},\\ Rn_y&=\frac{(\rho_p-\rho)(\phi_1-\phi_0)gk_yH}{\mu\alpha_m},\\ Rn_z&=-\frac{(\rho_p-\rho)(\phi_1-\phi_0)gk_zH}{\mu\alpha_m},\end{aligned}\right\} \quad \ldots(18)$$

$$N_A=\frac{D_T(T_h-T_C)}{D_BT_C(\phi_1-\phi_0)}, \quad \ldots(19)$$

and $$N_B=\frac{\varepsilon(\rho c)_p}{(\rho c)_f}(\phi_1-\phi_0). \quad \ldots(20)$$

3. BASIC SOLUTION:

We seek a time-independent quiescent solution of equations (9) – (14) with temperature and nanoparticle volume fraction varying in the z-direction only, which is a solution of the form

$$V=0,\ T=T_b(z),\ \phi=\phi_b(z).$$

Equation (11) and (12) reduces to

$$\frac{d^2T_b}{dz^2}+\frac{N_B}{L_e}\frac{d\phi_b}{dz}\frac{dT_b}{dz}+\frac{N_A N_B}{L_e}\left(\frac{dT_b}{dz}\right)^2=0\,, \qquad \ldots(21)$$

$$\frac{d^2\phi_b}{dz^2}+N_A\frac{d^2T_b}{dz^2}=0\,. \qquad \ldots(22)$$

Using the boundary conditions (13) and (14), equation (22) may be integrated to give

$$\phi_b=-N_A T_b+\left(1-N_A\right)z+N_A \qquad \ldots(23)$$

and substitution of this equation (21) gives

$$\frac{d^2T_b}{dz^2}+\frac{\left(1-N_A\right)N_B}{L_e}\frac{dT_b}{dz}=0 \qquad \ldots(24)$$

The solution of equation (24) satisfying equation (13) and (14) is

$$T_b=\frac{1-e^{-(1-N_A)N_B(1-z)/L_e}}{1-e^{-(1-N_A)N_B/L_e}}\,. \qquad \ldots(25)$$

The remainder of the basic solution is easily obtained by first substituting in equation (23) to obtain ϕ_b and then using integration of equation (10) to obtain p_b.

According to **Buongiorno [1]**, for most nanofluids investigated so far L_e is large, of the order of 10^5-10^6, while N_A is not greater than about 10. Then the exponents in equation (25) are small and so to a good approximation, one has

$$T_b=1-z\,, \qquad \ldots(26)$$

and so that

$$\phi_b=z\,. \qquad \ldots(27)$$

4. PERTURBATION SOLUTION:

We now superimpose perturbations on the basic solution, we write

$$v=v'\,,$$

$$\left(u',v',w'\right)=\left(u,v,w\right),\ P=P_b+P',\ T=T_b+T',\ \phi=\phi_b+\phi', \qquad \ldots(28)$$

Substitute in equation (9) to (14), and linearize by neglecting the products of primed quantities. The following equations are obtained when equations (26) and (27) are used.

$$\nabla.v'=0\,, \qquad \ldots(29)$$

$$\left.\begin{aligned}\frac{Da_x}{p_r}\frac{\partial u'}{\partial t}&=-\nabla p'+Da_x\nabla^2u',\\ \frac{Da_y}{p_r}\frac{\partial v'}{\partial t}&=-\nabla p'+Da_y\nabla^2v',\\ \frac{Da_z}{p_r}\frac{\partial w'}{\partial t}&=-\nabla p'+Da_z\nabla^2w'-w'+Ra_zT'e_z^a-Rn_z\phi'\hat{e}_z,\end{aligned}\right\} \qquad \ldots(30)$$

$$\frac{\partial T'}{\partial t}-w'=\nabla^2T'+\frac{N_B}{L_e}\left(\frac{\partial T'}{\partial z}-\frac{\partial \phi'}{\partial z}\right)-\frac{2N_AN_B}{L_e}\frac{\partial T'}{\partial z}\,, \qquad \ldots(31)$$

$$\frac{1}{\sigma}\frac{\partial \phi'}{\partial t}+\frac{1}{\varepsilon}w'=\frac{1}{L_e}\nabla^2\phi'+\frac{N_A}{L_e}\nabla^2T'\,, \qquad \ldots(32)$$

$$w' = 0\,,\ \frac{\partial w}{\partial z} + \lambda_1 \frac{\partial^2 w}{\partial z^2} = 0\,,\ T' = 0\,,\ \phi' = 0 \ \text{ at } \ z = 0\,, \qquad \ldots(33)$$

$$w' = 0\,,\ \frac{\partial w}{\partial z} - \lambda_2 \frac{\partial^2 w}{\partial z^2} = 0\,,\ T' = 0\,,\ \phi' = 0 \ \text{ at } \ z = 1. \qquad \ldots(34)$$

It will be noted that the parameter $\left(Rm_x, Rm_y, Rm_z\right)$ is not involved in these and subsequent equations. It is just a measure of the basic static pressure gradient.

For the case of a regular fluid (not a nanofluid) the parameters $\left(Rm_x, Rm_y, Rm_z\right)$, N_A and N_B are zero, the second term in equation (32) is absent because $d\phi_b/dz = 0$ rather than 1, and then equation (32) is satisfied trivially. The remaining equations are reduced to the familiar equations for the Brinkman extension of the Horton-Roger-Lapwood problem.

The six unknowns u', v', w', p', T' and ϕ' can be reduced to three by operating on equation (30) with $\hat{e}_z$ curlcurl and using equation (29). The result is

$$\left.\begin{aligned} &\frac{Da_x}{p_r}\frac{\partial}{\partial t}\nabla^2 u' - Da_x \nabla^4 u' = 0, \\ &\frac{Da_y}{p_r}\frac{\partial}{\partial t}\nabla^2 v' - Da_y \nabla^4 v' = 0, \\ &\frac{Da_z}{p_r}\frac{\partial}{\partial t}\nabla^2 w' - Da_z \nabla^4 w' + \nabla^2 w' = Ra_z \nabla_H^2 T' - Rn_z \nabla_H^2 \phi'. \end{aligned}\right\} \qquad \ldots(35)$$

Here ∇_H^2 is the two dimensional Laplacian operator on the horizontal plane.

The differential equations (31), (32) and (35) the boundary conditions (33) and (34) constitute a linear boundary-value problem that can be solved using the method of normal modes.

We write

$$\left(u', v', w', T', \phi'\right) = \left[u(z), v(z), w(z), \theta(z), \phi(z)\right] \exp\left(st + ilx + imy\right) \qquad \ldots(36)$$

and substitute into the differential equation to obtain

$$\left[Da^*\left(D^2-\alpha^2\right)D^2W - Da_z\left(D^2-\alpha^2\right)\alpha^2 W - \left(1+\frac{SDa^*}{P_r}\right)D^2W + \left(1+\frac{SDa_z}{P_r}\right)\alpha^2 W\right]$$

$$-Ra_z\alpha^2\theta + Rn_z\alpha^2\phi = 0\,, \qquad \ldots(37)$$

$$W + \left(D^2 + \frac{N_B}{L_e}D - \frac{2N_A N_B}{L_e}D - \alpha^2 - S\right)\theta - \frac{N_B}{L_e}D\phi = 0\,, \qquad \ldots(38)$$

$$\frac{1}{\varepsilon}W - \frac{N_A}{L_e}\left(D^2-\alpha^2\right)\theta - \left(\frac{1}{L_e}\left(D^2-\alpha^2\right) - \frac{S}{\sigma}\right)\phi = 0\,, \qquad \ldots(39)$$

$$W = 0\,,\ DW + \lambda_1 D^2 W = 0\,,\ \theta = 0\,,\ \phi = 0 \ \text{ at } \ z = 0\,, \qquad \ldots(40)$$

$$W = 0\,,\ DW - \lambda_2 D^2 W = 0\,,\ \theta = 0\,,\ \phi = 0 \ \text{ at } \ z = 1\,, \qquad \ldots(41)$$

where $D = \dfrac{d}{dz}$ and $\alpha = \left(l^2 + m^2\right)^{1/2}$. $\qquad \ldots(42)$

Thus, α is a dimensionless horizontal wave number.

For neutral stability, the real part of s is zero. Hence, we now write $s = i\omega$, where ω is real and is a dimensionless frequency.

We now employ a Galerkin-type weighted residuals method to obtain and approximate solution to the system of equations (37) – (41), we choose as trial functions (satisfying the boundary conditions) $W_p, \theta_p, \phi_p; p = 1, 2, 3, ...$, and write

$$\left.\begin{aligned} w &= \sum_{p=1}^{N} A_p w_p, \\ \theta &= \sum_{p=1}^{N} B_p \theta_p, \\ \phi &= \sum_{p=1}^{N} C_p \phi_p, \end{aligned}\right\} \qquad \text{...(43)}$$

Substitute into equations (37) – (39) and make the expression on the left-hand sides of those equations (the residuals) orthogonal to the trial functions, thereby obtaining a system of 3N linear algebraic equations in the 3N unknowns $A_p, B_p, C_p; p = 1, 2, ..., N$. The vanishing of the determinant of coefficient produces the eigenvalue equation for the system. One can regard Ra as the eigenvalue. Thus, Ra is found in terms of the other parameters.

5. RESULT AND DISCUSSION:

Free-Free boundaries

For this case, the boundary conditions are

$$W = 0,\ D^2W = 0,\ \theta = 0,\ \phi = 0 \ \text{at}\ z = 0 \ \text{and}\ z = 1, \qquad \text{...(44)}$$

and the trial functions can be chosen as

$$W_p = \theta_p = \phi_p = \sin p\pi z;\ p = 1, 2, 3, ... \qquad \text{...(45)}$$

Non-Oscillatory Convection

First, we consider the case of non-oscillatory instability, when $\omega = 0$.

For a first approximation we take N = 1. This produces the result

$$Ra_z = \frac{\left(Da^*\pi^2 + Da_z\alpha^2\right)\left(\pi^2+\alpha^2\right)^2 + \left(\pi^2+\alpha^2\right)}{\alpha^2} - \left(\frac{L_e}{\varepsilon} + N_A\right)Rn_z. \qquad \text{...(46)}$$

For the case $Da^* = Da_z = 0$, the minimum is allowed with $\alpha = \pi$ and minimum value is

$$Ra_z = 4\pi^2 - \left(\frac{L_e}{\varepsilon} + N_A\right)Rn_z. \qquad \text{...(47)}$$

On other hand in the case where Da^* and Da_z is large compared with unity, the minimum being attained at $\alpha = \frac{\pi}{\sqrt{2}}$, and the minimum value is

$$Ra_z = \frac{9\pi^4}{2}\left(Da^* + \frac{Da_z}{2}\right) - \left(\frac{L_e}{\varepsilon} + N_A\right)Rn_z. \qquad \text{...(48)}$$

One recognizes that in the absence of nanoparticles, one recovers the well known results that the critical Rayleigh-Darcy number is equal to $4\pi^2$ when $Da^* = 0, Da_z = 0$ and that the critical value of the fluid Rayleigh number is

$9\pi^4/2$ in the case where Da^* and Da_z tends to infinity. Usually when one employs a single term Galerkin approximation in this context, one gets an overestimate; however, in this case, the approximation happens to give the exact result.

As noted above, for a typical nanofluid, L_e is of the order of $10^5 - 10^6$ and N_A is not much > 10. Hence, the coefficient of Rn_z in equation (46) is large and negative. Thus, under the approximations we have made so far, we have the result that the presence of nanoparticles lowers the value of the critical Rayleigh number, usually by a substantial amount, in the case when Rn_z is positive, i.e. when the basic nanoparticle distribution is a top-heavy one.

It will be noted that in equation (46), the parameter N_B does not appear. The instability is almost purely a phenomenon due to buoyancy coupled with the conservation of nanoparticles. It is independent of the contributions of Brownian motion and thermophoresis to the thermal energy equation. Rather, the Brownian motion and thermophoresis enter to produce their effects directly into the equation expressing the conservation of nanoparticles, so that the temperature and the particle density are coupled in a particular way, and that results in the thermal and concentration buyoncy effects being coupled in the same way. It is useful to emphasize this by rewriting equation (46) in the form

$$Ra_z + \left(\frac{L_e}{\varepsilon} + N_A\right) Rn_z = \frac{\left(Da^*\pi^2 + Da_z\alpha^2\right)\left(\pi^2+\alpha^2\right)^2 + \left(\pi^2+\alpha^2\right)^2}{\alpha^2}, \qquad \ldots(49)$$

and noting that the left-hand side is a linear combination of the thermal Rayleigh number Ra_z and the concentration Rayleigh number Rn_z. The problem is analogous to the familiar double diffusive problem **(Nield and Bejan [6])**. It is also analogous to a bioconvection problem discussed in **Kuznetsov and Avramenko [4]**. We have defined Rn_z in a way so that it is positive when the applied particle density increases upward (the destabilizing situation). We note that Ra_z takes a negative value when Rn_z is sufficiently large. In this case, the destabilizing effect of concentration is so great that the bottom of the fluid layer must be cooled relative to the top to produce a state of neutral stability.

6. OSCILLATORY CONVECTION

We now consider the case $\omega \neq 0$ so $S \neq 0$; (since $S = i\omega$). We confine ourselves to the one-term Galerkin approximation. The eigenvalue equation now takes the form

$$Ra_z\alpha^2\left(\frac{J}{L_e} + \frac{i\omega}{\sigma}\right) + Rn_z\alpha^2\left(\frac{N_A}{L_e}J + \frac{(J+i\omega)}{\varepsilon}\right)$$

$$= \left[\left(Da^*\pi^2 + Da_z\alpha^2\right)J + J + \left(i\omega\frac{Da^*}{P_r}\right)\pi^2 + \left(i\omega\frac{Da_z}{P_r}\right)\alpha^2\right]\left(J + iw\right)\left(\frac{J_2}{L_e} + \frac{iw}{\sigma}\right), \qquad \ldots(50)$$

where $J = \pi^2 + \alpha^2$ …(51)

The real and imaginary parts of equation (50) yield

$$Ra_z + Rn_zN_A + Rn_z\frac{L_e}{\varepsilon}$$

$$= \left(Da^*\pi^2 + Da_z\alpha^2\right)\frac{J^2}{\alpha^2} + \frac{J^2}{\alpha^2} - \frac{\omega^2}{\alpha^2}\left(Da^*\pi^2 + Da_z\alpha^2\right)\frac{L_e}{\alpha} + \frac{L_e}{\sigma}$$

$$+\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\frac{L_{e}}{\sigma Pr}-\frac{1}{Pr}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right),$$

$$\Rightarrow Ra_{z}+Rn_{z}\left(N_{A}+\frac{L_{e}}{\varepsilon}\right)=\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}+1\right)\frac{J^{2}}{\alpha^{2}}-\frac{\omega^{2}}{\alpha^{2}}\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}+1\right)\frac{L_{e}}{\sigma}\right.$$

$$\left.+\frac{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)}{Pr}+\frac{L_{e}}{\sigma Pr}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right\}, \qquad \ldots(52)$$

and

$$Ra_{z}\frac{L_{e}}{\sigma}+Rn_{z}\frac{L_{e}}{\varepsilon}=-\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\frac{\omega^{2}L_{e}}{\sigma\alpha^{2}Pr}+\frac{J^{2}}{\alpha^{2}}\left[\frac{1}{Pr}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right.$$

$$\left.+\left(\frac{L_{e}}{\sigma}+1\right)\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right\}\right], \qquad \ldots(53)$$

elimination of ω^2 between equation (52) and (53), we get

$$Ra_{z}\left[\frac{L_{e}Pr}{\sigma}\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right\}+\frac{L_{e}}{\sigma}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right]$$

$$+Rn_{z}\left[\frac{L_{e}Pr}{\varepsilon}\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right\}+\left(\frac{\sigma}{\varepsilon}-N_{A}\right)\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right]$$

$$=\frac{J^{2}}{\alpha^{2}}\left[\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}+1\right)\left(\frac{L_{e}}{\sigma}+1\right)+\frac{1}{Pr}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right\}\right.$$

$$\left[Pr\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right\}+\left(1+\frac{\sigma}{L_{e}}\right)\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right]$$

$$\left.-\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}+1\right)\right]. \qquad \ldots(54)$$

One observes from equation (52) that in order for ω to be real, it is necessary that;

$$Ra_{z}+\left(\frac{L_{e}}{\varepsilon}+N_{A}\right)Rn_{z}\leq\frac{J^{2}\left[\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right]}{\alpha^{2}}. \qquad \ldots(55)$$

Hence, equation (54) gives the oscillatory stability boundary when equation (55) holds and the angular frequency ω of the oscillation is then given by

$$\omega^{2}=\frac{J^{2}\left[\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right]-Ra_{z}\alpha^{2}-\left(\frac{L_{e}}{\varepsilon}+N_{A}\right)Rn_{z}\alpha^{2}}{\left[\left\{\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+1\right\}\frac{L_{e}}{\sigma}+\frac{1}{Pr}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)+\frac{L_{e}}{\sigma Pr}\left(Da^{*}\pi^{2}+Da_{z}\alpha^{2}\right)\right]} \qquad \ldots(56)$$

7. CONCLUSION:

Here results have been obtained for non-oscillatory convection as well as oscillatory convection. The analysis predicts that oscillatory instability is possible in the case of a bottom heavy nanoparticle distribution.

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CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS WITH RESPECT TO (j, k) SYMMETRIC POINTS

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ABSTRACT:

We introduce a subclass of meromorphic analytic functions associated with the positive real part. Relationship with other well known classes such as meromorphic convex and starlike functions have been established. Further, very interesting integral representations have also been obtained.

Key words: ***meromorphic, multivalent,*** (j, k)**- *symmetrical functions*** .

AMS(MOS) ***2000 Subject Classifications*:30C45.**

1. Introduction, Definitions And Preliminaries

We let $\mathcal{H}(\mathcal{U})$ to denote the class of functions analytic in the open unit disc $\mathcal{U}$. For n a positive integer, we let

$$\mathcal{A}_n = \{f \in \mathcal{H},\, f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\}$$

and let $\mathcal{A} = \mathcal{A}_1$. The well known classes namely starlike, convex, close-to-convex and quasi-convex will be denoted by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ respectively. Our favorite references of the field are [1, 2] which covers most of the topics in a lucid and economical style. Let f and g be members $\mathcal{H}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $\mathcal{M}_p$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p}, \quad (p \geq 1), \tag{1}$$

which are analytic in the punctured open unit disk

$$\mathcal{U}^* = \{z : z \in \mathbb{C} \,\text{and}\, 0 <| z |< 1\} = \mathcal{U} \setminus \{0\}.$$

A function $f \in \mathcal{M}_p$ is said to be in the class $\mathcal{MS}^*(\gamma)$ of meromorphic starlike functions of order γ in $\mathcal{U}$ if and only if

$$Re\left(-\frac{1}{p}\frac{zf^{'}(z)}{f(z)}\right) > \gamma, \qquad (0 \leq \gamma < 1). \tag{2}$$

We let $\mathcal{MS}^* = \mathcal{MS}^*(0)$ to denote the meromorphic starlike functions.

A function $f \in \mathcal{M}_p$ is said to be in the class $\mathcal{MC}(\gamma)$ of meromorphic convex functions of order γ in $\mathcal{U}$ if and only if

$$Re\left(-\frac{1}{p}\frac{\left(zf^{'}(z)\right)^{'}}{f'(z)}\right) > \gamma, \qquad (0 \le \gamma < 1). \tag{3}$$

We let $\mathcal{MC} = \mathcal{MC}(0)$ to denote the meromorphic convex functions. We denote by $\mathcal{MK}(\gamma,\, \delta)$ and $\mathcal{MC}^*(\gamma,\, \delta)$, the respective multivalent meromorphic analogue of the close-to-convex and quasi-convex functions.

A function $f \in \mathcal{M}_p$ is said to be $(j,\, k)$-symmetrical if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon^{-pj} f(z), \tag{4}$$

$$(k = 1,\, 2,\, \ldots;\, j = 0,\, 1,\, 2,\, \ldots (k-1)).$$

The family of $(j,\, k)$-symmetrical functions will be denoted by $\mathcal{F}_k^j$. We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively. Also let $f_{j,\,k}(z)$ be defined by the following equality

$$f_{j,\,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^{\nu}z)}{\varepsilon^{-\nu pj}}, \tag{5}$$

$$(f \in \mathcal{M}_p;\, k = 1,\, 2,\, \ldots;\, j = 0,\, 1,\, 2,\, \ldots (k-1)).$$

It is obvious that $f_{j,\,k}(z)$ is a linear operator from $\mathcal{U}$ into $\mathcal{U}$. The notion of $(j,\, k)$-symmetric functions was introduced and studied by P. Liczberski and J. Połubiński in [4]. Sakaguchi in [5] introduced the notion of symmetric functions. Wang et. al. introduced the concept of starlike and convex functions with respect to k-symmetric points and studied interesting properties, see [6, 7, 8]. We now introduce the following classes of function in $\mathcal{M}_p$ with respect to $(j,\, k)$-symmetric points.

Definition 1.1. *A function $f \in \mathcal{M}_p$ is said to be in the class $\mathcal{MS}_p^{j,\,k}(\alpha,\, \beta)$ if it satisfies*

$$\alpha < Re\left\{-\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)}\right\} < \beta, \tag{6}$$

where $0 \le \alpha < 1 < \beta$ and $f_{j,\,k}(z) \ne 0$ in $\mathcal{U}^*$. Similarly, we call the class $\mathcal{MC}_p^{j,\,k}(\alpha,\, \beta)$ of functions $f \in \mathcal{M}_p$ with $f^{'}_{j,\,k}(z) \ne 0$ satisfying the subordination condition

$$\alpha < Re\left\{-\frac{1}{p}\frac{(zf'(z))^{'}}{f^{'}_{j,\,k}(z)}\right\} < \beta, \qquad (0 \le \alpha < 1 < \beta)\,. \tag{7}$$

Remark 1.2. *We observe that $f \in \mathcal{MS}_p^{j,\,k}(\alpha,\, \beta)$ satisfies each of the following subordination equations*

$$-\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} \prec \frac{1+(1-2\alpha)\,z}{1-z} \qquad \textit{and} \qquad -\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} \prec \frac{1+(1-2\beta)\,z}{1-z}.$$

Both superordinate functions in the above expression maps the unit disc onto right half plane, so it is obvious that $-\frac{1}{p}\frac{\left(zf'(z)\right)'}{f'_{j,\,k}(z)}$ *is mapped on to a plane having real part greater than* α *and but lesser than* β*. The classes* $\mathcal{MS}_p^{j,\,k}(\alpha,\,\beta)$ *and* $\mathcal{MC}_p^{j,\,k}(\alpha,\,\beta)$ *are defined in motivation by the class introduced by Kuroki and Owa in* [3].

Setting $\mathcal{G} = \left\{u + iv : \alpha < u < \beta\right\}$ with $p(z) = -\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)}$ or $p(z) = -\frac{1}{p}\frac{\left(zf'(z)\right)'}{f'_{j,\,k}(z)}$ and from the equivalent subordination condition proved by Kuroki and Owa in [3] for the class $\mathcal{S}(\alpha,\,\beta)$, we may rewrite the conditions (6) or (7) in the form

$$p(z) \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, z}{1-z}\right). \tag{8}$$

Further, we note that

$$h(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, z}{1-z}\right) \tag{9}$$

maps $\mathcal{U}$ onto a convex domain conformally and is of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \frac{\beta-\alpha}{n\pi}\, i\, \left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right)$.

We denote by $\mathcal{MK}_p(\alpha,\,\beta)$, the class of functions which satisfies the inequality

$$\alpha < Re\left\{-\frac{1}{p}\frac{zf'(z)}{g(z)}\right\} < \beta$$

where $g \in \mathcal{MS}^*$. Also, we let $\mathcal{MC}_p^*(\alpha,\,\beta)$ to denote the class of functions which satisfies the inequality

$$\alpha < Re\left\{-\frac{1}{p}\frac{(zf'(z))'}{g'(z)}\right\} < \beta$$

where $g \in \mathcal{MC}$.

2. **Integral Representations and Closure Properties**

We begin with the following.

Theorem 2.1. *Let* $f \in \mathcal{MS}_p^{j,\,k}(\alpha,\,\beta)$*, then we have*

$$f_{j,\,k}(z) = z^p \exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z}\frac{1}{t}\left[2 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(t)}{1-w(t)}\right)\right]\, dt\right\}, \tag{10}$$

where $f_{j\,k}(z)$ *defined by equality (5),* $w(z)$ *is analytic in* $\mathcal{U}$ *with* $w(0) = 0$ *and* $\mid w(z) \mid < 1$.

Proof. Let $f \in \mathcal{MS}_p^{j,\,k}(\alpha,\,\beta)$. In view of (8), we have

$$-\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} = 1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(z)}{1-w(z)}\right), \tag{11}$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0)=0$, $|\, w(z) \,|< 1$. Substituting z by $\varepsilon^{\nu} z$ in the equality (11) respectively ($\nu = 0,\, 1,\, 2,\, \ldots,\, k-1,\, \varepsilon^k = 1$), we have

$$-\frac{\varepsilon^{\nu} z f'(\varepsilon^{\nu} z)}{p\, f_{j,\,k}(\varepsilon^{\nu} z)} = 1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(\varepsilon^{\nu} z)}{1-w(\varepsilon^{\nu} z)}\right). \tag{12}$$

Using $f_{j,\,k}(\varepsilon^{\nu} z) = \varepsilon^{-\nu p j} f_{j,\,k}(z)$, equation (12) can be rewritten in the form

$$-\frac{z \varepsilon^{\nu+\nu p j} f'(\varepsilon^{\nu} z)}{p\, f_{j,\,k}(z)} = 1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(\varepsilon^{\nu} z)}{1-w(\varepsilon^{\nu} z)}\right). \tag{13}$$

Let $\nu = 0,\, 1,\, 2,\, \ldots,\, k-1$ in (13) respectively and summing them we get,

$$-\frac{z f'_{j,\,k}(z)}{p\, f_{j,\,k}(z)} = \frac{1}{k}\sum_{\nu=0}^{k-1} 1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(\varepsilon^{\nu} z)}{1-w(\varepsilon^{\nu} z)}\right). \tag{14}$$

From the equality (14), we get

$$\frac{f'_{j,\,k}(z)}{f_{j,\,k}(z)} - \frac{p}{z} = -\frac{p}{k\, z}\sum_{\nu=0}^{k-1} 1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(\varepsilon^{\nu} z)}{1-w(\varepsilon^{\nu} z)}\right) - \frac{p}{z}.$$

Since the presence of the first order pole at the origin, the difficulty to integrate the above equality is avoided by integrating from z_0 to z with $z_0 \neq 0$ and then let $z_0 \to 0$. We find

$$\begin{aligned}\log\left\{\frac{f_{j,\,k}(z)}{z^p}\right\} &= -\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[2 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(\varepsilon^{\nu}\zeta)}{1-w(\varepsilon^{\nu}\zeta)}\right)\right] d\zeta,\\ &= -\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^{\nu} z} \frac{1}{t}\left[2 + \frac{p\,(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(t)}{1-w(t)}\right)\right] dt,\end{aligned}$$

or equivalently,

$$f_{j,\,k}(z) = z^p \exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^{\nu} z} \frac{1}{t}\left[2 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(t)}{1-w(t)}\right)\right] dt\right\}.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $f \in \mathcal{MS}_p^{j,\,k}(\alpha,\, \beta)$, then we have*

$$\begin{aligned}f(z) = -p\int_0^z \zeta^{p-1}\exp&\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^{\nu} z} \frac{1}{t}\left[2 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(t)}{1-w(t)}\right)\right] dt\right\}\\ &\cdot\left[1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(\zeta)}{1-w(\zeta)}\right)\right] d\zeta\end{aligned} \tag{15}$$

where $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$ and $|\, w(z) \,|< 1$.

Proof. In view of (11), we have

$$\begin{aligned}z f'(z) &= -p\, f_{j,\,k}(z)\cdot\left[1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(z)}{1-w(z)}\right)\right]\\ &= -p z^p \exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^{\nu} z} \frac{1}{t}\left[2 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(t)}{1-w(t)}\right)\right] dt\right\}\\ &\quad\times\left[1 + \frac{(\beta-\alpha)}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\, w(z)}{1-w(z)}\right)\right]\end{aligned}$$

which upon integration establishes the Theorem 2.2. □

Theorem 2.3. *Let $f \in \mathcal{MC}_p^{j,\,k}(\alpha,\,\beta)$, then we have*

$$f_{j,\,k}(z) = \int_0^z \zeta^{p-1} \exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z}\left(\frac{1}{t}\left[1+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(t)}{1-w(t)}\right)\right]+\frac{1}{t}\right)dt\right\}d\zeta, \tag{16}$$

where $f_{j\,k}(z)$ defined by equality (5), $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$ and $\mid w(z)\mid < 1$.

Proof. Proceeding as in Theorem 2.5, we have

$$\frac{\left(zf'_{j,\,k}(z)\right)'}{zf'_{j,\,k}(z)} - \frac{p}{z} = -\frac{p}{k\,z}\sum_{\nu=0}^{k-1} 1+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(\varepsilon^\nu z)}{1-w(\varepsilon^\nu z)}\right)-\frac{p}{z}.$$

Integrating the above equality, we have

$$\begin{aligned}\log\left\{\frac{f'_{j,\,k}(z)}{z^{p-1}}\right\} &= -\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^z \frac{1}{\zeta}\left[2+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(\varepsilon^\nu\zeta)}{1-w(\varepsilon^\nu\zeta)}\right)\right]d\zeta\\ &= -\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z}\frac{1}{t}\left[2+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(t)}{1-w(t)}\right)\right]dt,\end{aligned}$$

Or equivalently,

$$f'_{j,\,k}(z) = z^{p-1}\exp\left\{\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu z}\frac{1}{t}\left[2+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(t)}{1-w(t)}\right)\right]dt\right\}.$$

Integrating the above equality completes the proof of the Theorem. □

Using the arguments similar to those detailed in Theorem 2.3, we can prove the following Theorem. We therefore, choose to omit the details involved.

Theorem 2.4. *Let $f \in \mathcal{MC}_p^{j,\,k}(\alpha,\,\beta)$, then we have*

$$\begin{aligned}f(z) &= -p\int_0^z \xi^{-1}\int_0^\xi \zeta^{p-1}\exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{\varepsilon^\nu\zeta} 2+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(t)}{1-w(t)}\right)dt\right\}\\ &\quad\times\left[1+\frac{(\beta-\alpha)}{\pi}i\log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}\,w(z)}{1-w(z)}\right)\right]d\zeta\,d\xi\end{aligned} \tag{17}$$

where $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$ and $\mid w(z)\mid < 1$.

Theorem 2.5. *Let $f \in \mathcal{MS}_p^{j,\,k}(\alpha,\,\beta)$, then $f \in \mathcal{MK}_p(\alpha,\,\beta)$.*

Proof. From the definition of $\mathcal{MS}_p^{j,\,k}(\alpha,\,\beta)$,

$$Re\left(-\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)}\right) > 0. \tag{18}$$

If we replace z by $\varepsilon^{\nu}z$ in (18), then (18) will be of the form

$$Re\left(-\frac{1}{p}\frac{\varepsilon^{\nu}zf'(\varepsilon^{\nu}z)}{f_{j,\,k}(\varepsilon^{\nu}z)}\right) > 0, \quad (z \in \mathcal{U}^{*};\ \nu = 0,\,1,\,2,\,\ldots,\,k-1). \tag{19}$$

It is obvious that $f_{j,\,k}(\varepsilon^{\nu}z) = \varepsilon^{-\nu pj}f_{j,\,k}(z)$ and $f'_{j,\,k}(\varepsilon^{\nu}z) = \varepsilon^{-\nu pj-\nu}f'_{j,\,k}(z)$. Using the equality in (19), we get

$$Re\left(-\frac{1}{p}\frac{\varepsilon^{\nu}zf^{'}(\varepsilon^{\nu}z)}{\varepsilon^{-\nu pj}f_{j,\,k}(z)}\right) > 0. \tag{20}$$

Let $\nu = 0,\,1,\,2,\,\ldots,\,k-1$ in (20) respectively and summing them, we get

$$Re\left(-\frac{1}{p}\,\frac{\sum_{\nu=0}^{k-1}\varepsilon^{\nu+\nu pj}zf'(\varepsilon^{\nu}z)}{f_{j,\,k}(z)}\right) > 0.$$

Or equivalently,

$$Re\left(-\frac{1}{p}\frac{z\,f'_{j,\,k}(z)}{f_{j,\,k}(z)}\right) > 0,$$

that is $f_{j,\,k}(z) \in \mathcal{S}^{*}$. Using this together with the condition (6) shows that $f \in \mathcal{MK}_{p}(\alpha,\,\beta)$. □

Theorem 2.6. *Let $f \in \mathcal{MC}_{p}^{j,\,k}(\alpha,\,\beta)$, then $f \in \mathcal{MC}_{p}^{*}(\alpha,\,\beta)$.*

Proof. Following the steps as in (2.5), we can establish $f \in \mathcal{MC}_{p}^{j,\,k}(\alpha,\,\beta)$ implies that $f_{j,\,k}(z) \in \mathcal{MC}$. Using this together with the condition (7) shows that $f \in \mathcal{MC}_{p}^{*}(\alpha,\,\beta)$. □

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THE STUDY OF SERIAL CHANNELS CONNECTED TO NON-SERIAL CHANNELS WITH FEEDBACK AND BALKING IN SERIAL CHANNELS AND RENEGING IN BOTH TYPES OF CHANNELS

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ABSTRACT :
A general queuing model having feedback, balking and reneging in serial queuing processes connected with non-serial queuing channels with reneging in random order selection for service has been studied in the present paper. Such models are of common occurrence in the administrative setup. The mean queue length of the model when queue discipline is first come first served is obtained for the model. Numerical results have also been obtained with respect to different types of customer's behaviour. Graphs representing mean queue length w.r.t. arrival rate and service rate variable have been obtained.
Keywords: ***Steady-State, difference-differential, waiting space, random selection, Poisson arrivals, exponential service, feedback, balking and reneging.***

1. INTRODUCTION:

Various researchers including O’brien (1954), Barrer (1955) and Finch (1959) studied the problems of serial queues in steady-state with Poisson assumption. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Singh (1994) worked on the network of queuing processes with impatient customers. Punam, et.al (2011) found the steady-state solution of serial queuing processes where feedback is not permitted. T.P. Singh etal (2014) explored the cost analysis of a queue model with inpatient customers. Satyabir, et.al (2014) obtained steady-state solution of serial queues with feedback, balking and reneging. However there may be situations where the serial queuing processes may be connected with non-serial queuing channels keeping the above observations in view, we in the present paper obtained the steady-state solutions for serial queuing processes with feedback, balking and reneging connected with non-serial queuing channels with reneging in which

(i) M-serial queuing processes with feedback, balking and reneging connected with N-non-serial queuing channels with reneging.
(ii) A customer may join any channel from outside and leave the system at any stage after getting service.
(iii) Feedback is permitted from each channel to its previous channel in serial channels.
(iv) The customer may balk due to long queue at each serial service channel.
(v) The impatient customer leaves both serial and non serial service channels after wait of certain time.
(vi) The input process in serial channels depends upon queue size and Poisson arrivals are followed.
(vii) Exponential service times are followed.
(viii) The queue discipline is random selection for service.
(ix) Waiting space is infinite.

The expressions for marginal probabilities and mean queue length have also been derived whenever the queue discipline is first come first served.

2. FORMULATION OF THE MODEL:

The system consists of the serial queues $Q_j\left(j=1,2,3,....,M\right)$ and non-serial channels $Q_{1i}(i=1,2,3,....N)$ with respective servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j\left(j=1,2,3,......,M\right)$ and $\lambda_{1i}\left(i=1,2,3,...N\right)$ at $Q_j\left(j=1,2,3,....,M\right)$ and $Q_{1i}(i=1,2,3,....N)$ but the sight of long queue at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$. Then the Poisson input rate at $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$ would be $\frac{\lambda_j}{n_j+1}$ where n_j is the queue size of $Q_j(\mathrm{j}=1,2,3,....\mathrm{M})$. Further, the impatient customer joining any serial service channel $Q_j\left(j=1,2,3,....,M\right)$ and non-serial channel $Q_{1i}(i=1,2,3,....N)$ may leave the queue without getting service after wait of certain time. Service time distributions for servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$ are mutually independent negative exponential distribution with parameters $\mu_j\left(j=1,2,......,M\right)$ and $\mu_{1i}\left(i=1,2,3,...N\right)$ respectively. After the completion of service at S_j, the customer either leaves the system with probability p_j or joins the next channel with probability $\frac{q_j}{n_{j+1}+1}$ or join back the previous channel with probability $\frac{r_j}{n_{j-1}+1}$ such that $p_j+\frac{q_j}{n_{j+1}+1}+\frac{r_j}{n_{j-1}+1}=1\ \left(j=1,2,3,...M-1\right)$ and after the completion of service at S_M the customer either leaves the system with probability p_M or join back the previous channel with probability $\frac{r_M}{n_{M-1}+1}$ or join any queue $Q_{1i}\left(i=1,2,3,.....N\right)$ with probability $q_{Mi}\left(i=1,2,3,....N\right)$ such that $p_M+\frac{r_M}{n_{M-1}+1}+\sum_{i=1}^{N}q_{Mi}=1$. It is being mentioned here that $r_j=0$ for $j=1$ as there is no previous channel of the first channel.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block Development officer, Tehsildar, Sub-Divisional Magistrate, District Magistrate etc. These officers correspond to the servers of serial channels. Education Department, Health Department, Irrigation Department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The senior officer may send any customer to his junior if some information regarding the customer’s problem is lacking. Further District Magistrate may send the customers to different departments such as Education, Health, Irrigation etc if there problems are related to such departments. The customer after seeing long queues before any serial service channel may decide not to enter the queue. It generally happens that person becomes impatient after joining the queue and may leave the channel without getting service.

3. FORMATION OF EQUATIONS:

Define: $P\left(n_1,n_2,n_3,.....n_{M-1},n_M,m_1,m_2,m_3,.....m_{N-1},m_N;t\right)$ = the probability that at time 't' there are n_j customers (which may balk, renege or leave the system after being serviced or join the next phase or join back the previous channel) waiting before $S_j\left(j=1,2,3,....,M-1,M\right)$; m_i customers(which may renege or leave the system after being serviced) waiting before the severs $S_{1i}\left(i=1,2,3,.....N\right)$.

We define the operators $T_i.,\ T._i,\ T._{i,i+1}.,T_{i-1}.,._i$ to act upon the vectors $\tilde{n}=\left(n_1,n_2,n_3,.....n_M\right)$ or $\tilde{m}=\left(m_1,m_2,m_3,.....m_N\right)$ as fallows

$$T_i.\left(\tilde{n}\right)=\left(n_1,n_2,n_3,.....,n_i-1,....n_M\right)$$

$$T._i\left(\tilde{n}\right)=\left(n_1,n_2,n_3,.....,n_i+1,....,n_M\right)$$

$$T._{i},_{i+1}.\left(\tilde{n}\right)=\left(n_1,n_2,n_3,.....,n_i+1,n_{i+1}-1,....,n_M\right)$$

$$T_{i-1}.,._i\left(\tilde{n}\right)=\left(n_1,n_2,n_3,.....,n_{i-1}-1,n_i+1....,n_M\right)$$

Following the procedure given by Kelly [5], we write the difference – differential equations as

$$\begin{aligned}
\frac{d\,P\left(\tilde{n},\tilde{m};t\right)}{dt} &= -\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta\left(n_i\right)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\lambda_{1j}+\sum_{j=1}^{N}\delta\left(m_j\right)\left(\mu_{1j}+D_{jm_j}\right)\right]P\left(\tilde{n},\tilde{m};t\right)\\
&+\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_i.\left(\tilde{n}\right),\tilde{m};t\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T._i\left(\tilde{n}\right),\tilde{m};t\right)\\
&+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T._{i},_{i+1}.\left(\tilde{n}\right),\tilde{m};t\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1}.,._i\left(\tilde{n}\right),\tilde{m};t\right).\\
&+\sum_{j=1}^{N}\mu_M q_{Mj}P\left(n_1,n_2,....n_M+1,T_j.\left(\tilde{m}\right);t\right)\\
&+\sum_{j=1}^{N}\lambda_{1j}P\left(\tilde{n},T_j.\left(\tilde{m}\right);t\right)+\sum_{j=1}^{N}\left(\mu_{1j}+D_{jm_{j+1}}\right)P\left(\tilde{n},T._j\left(\tilde{m}\right);t\right)
\end{aligned}\tag{3.1}$$

for $n_i\geq 0\ \left(i=1,2,3,....,M\right)$, $m_j\geq 0\left(j=1,2,3,...N\right)$;

where $\delta\left(x\right)=\begin{cases}1 & if\quad x\neq 0\\ 0 & if\quad x=0\end{cases}$ and $P\left(\tilde{n},\tilde{m};t\right)=\tilde{0}$ if any of the arguments in negative.

4. STEADY–STATE EQUATIONS:

We write the following Steady–state equations of the queuing model by equating the time-derivates to zero in the equation (2.1)

$$\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta(n_i)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\lambda_{1j}+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+D_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$
$$=\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_i.(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T._i(\tilde{n}),\tilde{m}\right)$$
$$+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T._{i,i+1}.(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1}.,._i(\tilde{n}),\tilde{m}\right).$$
$$+\sum_{j=1}^{N}\mu_M q_{Mj}P\left(n_1,n_2,....n_M+1,T_j.(\tilde{m})\right)$$
$$+\sum_{j=1}^{N}\lambda_{1j}P\left(\tilde{n},T_j.(\tilde{m})\right)+\sum_{j=1}^{N}\left(\mu_{1j}+D_{jm_{j+1}}\right)P\left(\tilde{n},T._j(\tilde{m})\right) \tag{4.1}$$

for $n_i\geq 0\ \ (i=1,2,3,....,M)$, $m_j\geq 0(\ j=1,2,3,...N)$;

5. STEADY-STATE SOLUTIONS:-

The solutions of the Steady-State equations (3.1) can be verified to be

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{\left(\lambda_1+\frac{\mu_2 r_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}\right)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right).$$
$$.\left(\frac{1}{n_2!}\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)}+\frac{\mu_3 r_3\rho_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)}\right)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right).$$
$$.\left(\frac{1}{n_3!}\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}+\frac{\mu_4 r_4\rho_4}{(n_4+1)\left(\mu_4+C_{4n_4+1}\right)}\right)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right).....$$

$$\cdot\left(\frac{1}{n_{M-1}!}\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{\left(n_{M-2}+1\right)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\frac{\mu_M r_M\rho_M}{\left(n_M+1\right)\left(\mu_M+C_{Mn_M+1}\right)}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}\left(\mu_{M-1}+C_{M-1i}\right)}\right)\cdot$$

$$\cdot\left(\frac{1}{n_M!}\frac{\left(\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{\left(n_{M-1}+1\right)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}\right)^{n_M}}{\prod_{i=1}^{n_M}\left(\mu_M+C_{Mi}\right)}\right)\left(\frac{\left(\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_1}}{\prod_{j=i}^{m_1}(\mu_{11}+D_{1j})}\right)\cdot$$

$$\cdot\left(\frac{\left(\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_2}}{\prod_{j=i}^{m_2}(\mu_{12}+D_{2j})}\right)\cdots\cdots\left(\frac{\left(\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_N}}{\prod_{j=i}^{m_1}(\mu_{1N}+D_{Nj})}\right) \tag{5.1}$$

$n_i \geq 0 \ \left(i=1,2,3,....,M\right)$, $m_j \geq 0\left(j=1,2,3,...N\right)$

where

$$\rho_1=\lambda_1+\frac{\mu_2 r_2\rho_2}{\left(n_2+1\right)\left(\mu_2+C_{2n_2+1}\right)}$$

$$\rho_2=\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{\left(n_1+1\right)\left(\mu_1+C_{1n_1+1}\right)}+\frac{\mu_3 r_3\rho_3}{\left(n_3+1\right)\left(\mu_3+C_{3n_3+1}\right)}\right)$$

$$\rho_3=\lambda_3+\frac{\mu_2 q_2\rho_2}{\left(n_2+1\right)\left(\mu_2+C_{2n_2+1}\right)}+\frac{\mu_4 r_4\rho_4}{\left(n_4+1\right)\left(\mu_4+C_{4n_4+1}\right)} \tag{5.2}$$

…

…

$$\rho_{M-1}=\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{\left(n_{M-2}+1\right)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\frac{\mu_M r_M\rho_M}{\left(n_M+1\right)\left(\mu_M+C_{Mn_M+1}\right)}$$

$$\rho_M = \lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1} + C_{M-1 n_{M-1}+1}\right)}$$

Solving these (4.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M = \frac{\left(\begin{array}{l} \lambda_M \Delta_{M-1} + \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} \lambda_{M-1}\Delta_{M-2} + \\ \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \lambda_{M-2}\Delta_{M-3} + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \ldots . \\ . \frac{q_3\mu_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)} \lambda_3\Delta_2 \\ + \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \ldots . \\ . \frac{q_3\mu_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)} . \frac{q_2\mu_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)} \lambda_2\Delta_1 \\ + \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} \ldots . \\ . \frac{q_3\mu_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)} . \frac{q_2\mu_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)} . \frac{q_1\mu_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)} \lambda_1 \end{array}\right)}{\left(\Delta_{M-1} - \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{r_M\mu_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)} \Delta_{M-2}\right)} \tag{5.3}$$

where $\Delta_M = \Delta_{M-1} - \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)} . \frac{r_M\mu_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)} \Delta_{M-2}$

where

$$\Delta_1 = 1, \qquad \Delta_2 = \begin{vmatrix} 1 & -\dfrac{\frac{r_2\mu_2}{n_2+1}}{\mu_2 + c_{2n_2+1}} \\ -\dfrac{\frac{q_1\mu_1}{n_1+1}}{\mu_1 + c_{1n_1+1}} & 1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 1 & -\dfrac{\frac{r_2\mu_2}{n_2+1}}{\mu_2 + c_{2n_2+1}} & 0 \\ -\dfrac{\frac{q_1\mu_1}{n_1+1}}{\mu_1 + c_{1n_1+1}} & 1 & -\dfrac{\frac{r_3\mu_3}{n_3+1}}{\mu_3 + c_{3n_3+1}} \\ 0 & -\dfrac{\frac{q_2\mu_2}{n_2+1}}{\mu_2 + c_{2n_2+1}} & 1 \end{vmatrix}$$

…

… (5.4)

$$\Delta_M = \begin{vmatrix} 1 & -\dfrac{\frac{r_2}{n_2+1}\mu_2}{\mu_2 + C_{2n_2+1}} & 0 & 0 & - & - & - & 0 & 0 & 0 \\ -\dfrac{\frac{q_1}{n_1+1}\mu_1}{\mu_1 + C_{1n_1+1}} & 1 & -\dfrac{\frac{r_3}{n_3+1}\mu_3}{\mu_3 + C_{3n_3+1}} & 0 & - & - & - & 0 & 0 & 0 \\ 0 & -\dfrac{\frac{q_2}{n_2+1}\mu_2}{\mu_2 + C_{2n_2+1}} & 1 & -\dfrac{\frac{r_4}{n_4+1}\mu_4}{\mu_4 + C_{4n_4+1}} & - & - & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & - & -\dfrac{\frac{q_{M-2}}{n_{M-2}+1}\mu_{M-2}}{\mu_{M-2} + C_{M-2n_{M-2}+1}} & 1 & -\dfrac{\frac{r_M}{n_M+1}\mu_M}{\mu_M + C_{Mn_M+1}} \\ 0 & 0 & 0 & 0 & - & - & - & 0 & -\dfrac{\frac{q_{M-1}}{n_{M-1}+1}\mu_{M-1}}{\mu_{M-1} + C_{M-1n_{M-1}+1}} & 1 \end{vmatrix}$$

Since ρ_M is obtained, so we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (4.2), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of (4.2). Continuing in this way, we shall obtain $\rho_{M-3}, \rho_{M-4}, ---, \rho_3, \rho_2$ and ρ_1.

Thus, we write (4.1) as under

$$p(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right)\left(\frac{1}{n_2!}\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right)\left(\frac{1}{n_3!}\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right)....$$

$$.\left(\frac{1}{n_{M-1}!}\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right).\left(\frac{1}{n_M!}\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right).$$

$$.\quad .\left(\frac{\left(\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_1}}{\prod_{j=i}^{m_1}(\mu_{11}+D_{1j})}\right)\left(\frac{\left(\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_2}}{\prod_{j=i}^{m_2}(\mu_{12}+D_{2j})}\right).$$

$$.....\left(\frac{\left(\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_N}}{\prod_{j=i}^{m_1}(\mu_{1N}+D_{Nj})}\right) \tag{5.5}$$

$$n_i \geq 0 \ (i=1,2,3,....,M), m_j \geq 0 (j=1,2,3,...N)$$

We obtain $P(\tilde{0},\tilde{0})$ from the normalizing conditions.

$$\sum_{\tilde{n}=\tilde{0},\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n},\tilde{m}) = 1 \tag{5.6}$$

and with the restriction that traffic intensity of each service channel of the system is less than unity, C_{in_i} and D_{jm_j} are reneging rates at which customer renege after a wait of time T_{0i} whenever there are n_i and m_j customer in the service channels Q_i and Q_{1j}.

$$C_{in_i} = \frac{\mu_{1i} e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}{1-e^{-\frac{\mu_{1i}T_{0i}}{n_i}}} \qquad (i=1,2,3,....M) \text{ and}$$

$$D_{jm_j} = \frac{\mu_{1j} e^{-\frac{\mu_{1j} T_{0j}}{m_j}}}{1 - e^{-\frac{\mu_{1j} T_{0j}}{m_j}}} \quad (j = 1, 2, 3, ...N)$$

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive. Putting $C_{in_i} = C_i \quad (i = 1, 2, 3, ...M)$ and $D_{jm_j} = D_j \quad (j = 1, 2, 3, ...N)$ in the steady-state solution (4.1) then $\rho_i \left(i = 1, 2, 3, ...\text{M}\right)$ will change accordingly and the steady- state solution reduces to

$$P(\tilde{n},\tilde{\text{m}}) = P(\tilde{0},\tilde{0}) \left(\frac{1}{n_1!} \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \right) \left(\frac{1}{n_2!} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} \right) \left(\frac{1}{n_3!} \left(\frac{\rho_3}{\mu_3 + C_3} \right)^{n_3} \right) ....$$

$$\cdot \left(\frac{1}{n_{M-1}!} \left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right)^{n_{M-1}} \right) \left(\frac{1}{n_M!} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M} \right).$$

$$\cdot \left(\frac{\lambda_{11} + \frac{\mu_M q_{M1} \rho_M}{(n_M + 1)(\mu_M + C_M)}}{(\mu_{11} + D_1)} \right)^{m_1} \left(\frac{\lambda_{12} + \frac{\mu_M q_{M2} \rho_M}{(n_M + 1)(\mu_M + C_M)}}{(\mu_{12} + D_2)} \right)^{m_2} \cdots \left(\frac{\lambda_{1N} + \frac{\mu_M q_{MN} \rho_M}{(n_M + 1)(\mu_M + C_M)}}{(\mu_{1N} + D_N)} \right)^{m_N} \tag{5.7}$$

We obtain $P\left(\tilde{0},\tilde{0}\right)$ from (4.6) and (4.7) as

$$\left(P\left(\tilde{0},\tilde{0}\right)\right)^{-1} = \prod_{i=1}^{M} e^{\frac{\rho_i}{\mu_i + C_i}} \prod_{j=1}^{N} \frac{1}{1 - \rho_{1j}}$$

where $$\rho_{1j} = \frac{\lambda_{1j} + \frac{\mu_M q_{Mj} \rho_M}{(n_M + 1)(\mu_M + C_M)}}{(\mu_{1j} + D_j)}, \qquad j = 1, 2, 3, ...N$$

Thus $P\left(\tilde{n},\tilde{m}\right)$ is completely determined.

6. STEADY-STATE MARGINAL PROBABILITIES:

Let $P\left(n_1\right)$ be the steady-state marginal probability that there are n_1 units in the queue before the first server. This is determined as

$$P(n_1) = \sum_{n_2,n_3,....n_M=0}^{\infty} \sum_{\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n},\tilde{m})$$

$$= \sum_{n_2,n_3,....n_M=0}^{\infty} \sum_{\tilde{m}=\tilde{0}}^{\infty} P(\tilde{0},\tilde{0}) \left(\frac{1}{n_1!} \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \right) \left(\frac{1}{n_2!} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} \right) \left(\frac{1}{n_3!} \left(\frac{\rho_3}{\mu_3 + C_3} \right)^{n_3} \right) ...$$

$$\left(\frac{1}{n_{M-1}!} \left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right)^{n_{M-1}} \right) \left(\frac{1}{n_M!} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M} \right) .(\rho_{11})^{m_1} (\rho_{12})^{m_2} ...(\rho_{1N})^{m_N}$$

Thus $P(n_1) = \frac{1}{n_1!} \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} e^{-(\frac{\rho_1}{\mu_1 + C_1})} \qquad n_1 > 0$

Similarly

$$P(n_2) = \frac{1}{n_2!} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} e^{-(\frac{\rho_2}{\mu_2 + C_2})} \qquad n_2 > 0$$

…

…

$$P(n_M) = \frac{1}{n_M!} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M} e^{-(\frac{\rho_M}{\mu_M + C_M})} \qquad n_M > 0$$

Further let $P(m_1), P(m_2), P(m_3),....,P(m_N)$ be the steady-state marginal probabilities that there are $m_1, m_2, m_3,.....,m_N$ customers waiting before server $S_{1i}(i = 1,2,3,....,N)$ respectively.

$$P(m_1) = \sum_{\tilde{n}=0}^{\infty} \sum_{m_2,m_3,...m_N=0}^{\infty} P(\tilde{n},\tilde{m})$$

$$= \sum_{\tilde{n}=0}^{\infty} \sum_{m_2,m_3,...m_N=0}^{\infty} P(\tilde{0},\tilde{0}) \left(\frac{1}{n_1!} \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \right) \left(\frac{1}{n_2!} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} \right) \left(\frac{1}{n_3!} \left(\frac{\rho_3}{\mu_3 + C_3} \right)^{n_3} \right) ....$$

$$\left(\frac{1}{n_{M-1}!} \left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right)^{n_{M-1}} \right) \left(\frac{1}{n_M!} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M} \right) .(\rho_{11})^{m_1} (\rho_{12})^{m_2} ...(\rho_{1N})^{m_N}$$

$$= (\rho_{11})^{m_1} (1 - \rho_{11}) \qquad m_1 > 0$$

Similarly

$$P(m_2) = (\rho_{12})^{m_2} (1 - \rho_{12}) \qquad m_2 > 0$$

…

$$P(m_N) = (\rho_{1N})^{m_N} (1 - \rho_{1N}) \qquad m_N > 0$$

7. MEAN QUEUE LENGTH:

Mean queue length before the server S_1 is determined by

$$L_1 = \sum_{n_1=0}^{\infty} n_1 P(n_1) = \sum_{n_1=0}^{\infty} n_1 \frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1} e^{-(\frac{\rho_1}{\mu_1+C_1})}$$

$$= \frac{\rho_1}{\mu_1 + C_1}$$

Similarly

$$L_2 = \frac{\rho_2}{\mu_2 + C_2}$$

…

…

$$L_M = \frac{\rho_M}{\mu_M + C_M}$$

Mean queue length before the server S_{11} is determined as

$$L_{11} = \frac{\rho_{11}}{1 - \rho_{11}}$$

Similarly

$$L_{1j} = \frac{\rho_{1j}}{1 - \rho_{1j}} \qquad j = 2,3,....,N$$

Hence mean queue length of the system is

$$L = \sum_{k=1}^{M} L_k + \sum_{j=1}^{N} L_{1j}$$

8. Numerical Solutions of Marginal Mean Queue Length of Model:

Serial /Non-Serial Servers S_i/S_{1i}	Arrival rate λ_i before server S_i	Reneging rate C_i before server S_i	Service rate μ_i before server S_i	Prob. of joining the next server $q_i/(n_i+1)$	Prob. of joining the previous server $r_i/(n_i+1)$	ρ_M	Mean queue length before the server S_i $L_i = \rho_i/(\mu_i+C_i)$	Arrival rate λ_{1i} before servers S_{1i}	Prob. of joining the non-serial servers q_{Mi}	Service rate μ_{1i} before servers S_{1i}	Reneging rate in non-serial channels D_i	ρ_{1i}	Mean Queue Length before the servers S_{1i} $=\rho_{1i}/(1-\rho_{1i})$	Sum of Mean Queue Lengths of Serial and Non-Serial Servers
1	12	1	14	0.0625	0	5.3861	0.35907	16	0.03	18	2	0.802	4.047017	4.40609
2	14	2	15	0.04	0.03	14.671	0.86299	15	0.04	16	3	0.792	3.809742	4.67274
3	12	3	13	0.016667	0.033333	13.171	0.82321	13	0.06	15	1	0.817	4.469200	5.29242
4	15	1	16	0.036364	0.045455	15.28	0.89882	14	0.07	16	2	0.783	3.599989	4.49881
5	16	3	17	0.05	0.007143	16.742	0.83711	15	0.09	17	1	0.84	5.232233	6.06935
6	18	2	19	0.0375	0.0125	19.391	0.92336	17	0.08	19	3	0.777	3.489217	4.41259
7	17	4	18	0.023077	0.046154	17.986	0.81754	12	0.01	15	1	0.751	3.012459	3.83001
8	19	1	20	0.029412	0.017647	19.526	0.92982	14	0.07	16	2	0.783	3.599989	4.52982
9	21	2	22	0.072727	0.009091	22.416	0.93401	19	0.02	21	3	0.793	3.823967	4.75799
10	23	4	25	-	0.041176	24.494	0.84463	20	0.05	22	4	0.772	3.378656	4.22329
Sum of Mean Queue Length of the Model =46.69309909														

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