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ON GENERALIZED *q*-DIFFERENTIAL TRANSFORM

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ABSTRACT
In the present paper, we first establish generalized q-Taylor's formula involving Caputo fractional q-derivatives. Next, we define generalized q-differential transform and its inverse and establish some basic properties for this transform. Finally, we apply this transform to solve some linear and nonlinear fractional q-difference equations with Caputo fractional q-derivatives.
Keywords: q-Taylor's formula, Caputo fractional q-derivatives, q-differential transform method, fractional q-difference equations.
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1. INTRODUCTION

Fractional calculus is a generalization of ordinary calculus to arbitrary order. Mathematical modeling of many physical systems leads to linear and nonlinear fractional differential equations in various fields of physics and engineering. Recently, various analytical and numerical methods have been employed to solve linear and non-linear fractional differential equations. One such method developed by Odibat *et al.* [15] is the generalized differential transform method for fractional differential equations involving Caputo fractional derivative [5]. This method is a generalization of differential transform method introduced by Zhou [7].

The *q*-calculus has in the last twenty years served as a bridge between mathematics and physics. The *q*-difference calculus or quantum calculus was initially developed by Jackson [1, 3]. The fractional *q*-difference calculus had its origin in the works of Al-Salam [4] and Agarwal [6], whereas definition of Caputo type fractional *q*-derivative is given in [16]. The *q*-deformation of nonlinear integrable differential equations is started by Frenkel [9]. The *q*-differential transform was introduced by Jing and Fan [8] for solving ordinary *q*-difference equations.

In this paper, we present a new generalization of *q*-differential transform that will extend the application of the transform to solve fractional *q*-difference equations. The new transform will be named as generalized *q*-differential transform. For this, we first establish generalized *q*-Taylor's formula involving Caputo fractional *q*-derivative, which for $q \to 1$ reduces to the generalized Taylor's formula [13].

The remaining paper is organized as follows. In Section 2, we provide definitions which will be used in subsequent sections. In Section 3, we establish generalized *q*-Taylor's formula involving Caputo fractional *q*-derivatives. In Section 4, we define generalized *q*-differential transform and its inverse and establish some basic properties for this transform. In Section 5, we solve some linear and nonlinear fractional *q*-difference equations with Caputo fractional *q*-derivatives.

2. DEFINITIONS

For $q \in \mathbb{C}$, $0 < |q| < 1$, ***q*-analogue of a real number** α is given by [1.1]

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}. \tag{1.1}$$

The ***q*-shifted factorial** (*q*-analogue of the Pochhammer Symbol) is defined by

$$(a;q)_k=\begin{cases}\prod_{j=0}^{k-1}\left(1-aq^j\right) & \text{if } k>0\\ 1 & \text{if } k=0\\ \prod_{j=0}^{\infty}\left(1-aq^j\right) & \text{if } k\to\infty\end{cases},$$

or equivalently

$$(a;q)_k=\frac{(a;q)_\infty}{\left(aq^k;q\right)_\infty},\ (k\in N),$$

and for any complex number α ,

$$(a;q)_\alpha=\frac{(a;q)_\infty}{\left(aq^\alpha;q\right)_\infty}, \tag{1.2}$$

where the principal value of q^α is taken.

***q*-analogue of power function** [11]

$$(a-b)_q^\alpha=a^\alpha\left(b/a;q\right)_\alpha$$

$$=\alpha^\alpha\prod_{j=0}^{\infty}\left[\frac{1-\left(b/a\right)q^j}{1-\left(b/a\right)q^{j+\alpha}}\right]=a^\alpha\frac{\left(b/a;q\right)_\infty}{\left(q^\alpha b/a;q\right)_\infty},(a\neq 0), \tag{1.3}$$

and

$$(a-b)_q^\alpha=(a-b)(a-bq)_q^{\alpha-1}. \tag{1.4}$$

The ***q*-gamma** function is defined by [11]

$$\Gamma_q(\alpha)=(1-q)_q^{\alpha-1}(1-q)^{1-\alpha};R(\alpha)>0. \tag{1.5}$$

Second mean value theorem for *q*-integral [10]

If $f(x)$ and $g(x)$ are continuous functions on $[a,b]$ and $g(x)\geq 0$ for $x\in[a,b]$, then $\exists c\in(a,b)$ and $\hat{q}\in(0,1)$ such that

$$\int_a^b f(t)g(t)d_qt=f(c)\int_a^b g(t)d_qt \qquad \forall q\in(\hat{q},1). \tag{1.6}$$

***q*-Mittag-Leffler function [17]**

For $\Re\ (\alpha)>0;\ \beta\in\Re^+$, the q-Mittag-Leffler functions are defined by

$${}_qE_\alpha(\lambda,x-a)=\sum_{n=0}^{\infty}\lambda^k\frac{(x-a)_q^{\alpha k}}{\Gamma_q(\alpha k+1)}, \tag{1.7}$$

and

$${}_qE_{\alpha,\beta}(\lambda,x-a)=\sum_{k=0}^{\infty}\lambda^k\frac{(x-a)_q^{\alpha k}}{\Gamma_q(\alpha k+\beta)}. \tag{1.8}$$

q-analogue of generalized Mittag-Leffler function [17]

For $\alpha, l \in \mathbb{C}$ and $m \in \Re$, the generalized q-Mittag-Leffler function is defined by

$$ {}_qE_{\alpha,m,l}\left(\lambda, x-a\right)=\sum_{k=0}^{\infty}\lambda^k c_k q^{-\frac{k(k-1)}{2}\alpha^2(m-1)(l+1)}\left(x-a\right)_q^{\alpha mk}, \tag{1.9}$$

where

$$c_0=1 \text{ and } c_k=\prod_{j=0}^{k-1}\frac{\Gamma_q\left[\alpha\left(jm+l\right)+1\right]}{\Gamma_q\left[\alpha\left(jm+l+1\right)+1\right]}, k=1,2,3,..., \tag{1.10}$$

with $\Re\left(\alpha\right)>0, m>0$ and $\alpha\,(jm+l)\notin \boldsymbol{Z}^-$ $(j\in \boldsymbol{N}_0)$

In particular

$$ {}_qE_{\alpha,1,l}\left(\lambda, x-a\right)=\Gamma_q\left(\alpha l+1\right){}_qE_{\alpha,\alpha l+1}\left(\lambda, x-a\right). \tag{1.11}$$

The Riemann-Liouville fractional q-integral operator, is defined as [6]

$$I_{q,a}^{\alpha}f\left(x\right)=\frac{1}{\Gamma_q\left(\alpha\right)}\int_a^x\left(x-tq\right)_q^{\alpha-1}f\left(t\right)d_qt;\ \alpha\in\mathbb{R}^+,\ a<x, \tag{1.12}$$

where q-integral is defined as follows :

$$\int_0^x f\left(t\right)d_qt=x\left(1-q\right)\sum_{n=0}^{\infty}q^nf\left(xq^n\right), \tag{1.13}$$

and

$$\int_a^x f\left(t\right)d_qt=\int_0^x f\left(t\right)d_qt-\int_0^a f\left(t\right)d_qt, \tag{1.14}$$

For $\alpha,\beta\in\Re^+$, the fractional q-integration has the following semi group property [6]

$$I_{q,a}^{\beta}I_{q,a}^{\alpha}f\left(x\right)=I_{q,a}^{\alpha+\beta}f\left(x\right),\ a<x. \tag{1.15}$$

For $\alpha\in\Re^+$, we have [14]

$$I_{q,a}^{\alpha}\left(x-a\right)_q^{\lambda}=\frac{\Gamma_q\left(\lambda+1\right)}{\Gamma_q\left(\lambda+\alpha+1\right)}\left(x-a\right)_q^{\lambda+\alpha}, a<x, \lambda+1>0. \tag{1.16}$$

The **Caputo type fractional q-derivative**, for $n-1<\alpha\le n$ is defined as [16]

$$ {}_*D_{q,a}^{\alpha}f\left(x\right)=I_{q,a}^{n-\alpha}D_q^nf\left(x\right),\ n-1<\alpha\le n,\ n\in\mathbb{N}. \tag{1.17}$$

where

$$\left(D_qf\right)\left(x\right)=\frac{f\left(x\right)-f\left(qx\right)}{\left(1-q\right)x},\ \left(x\neq 0\right), \tag{1.18}$$

when $a=0$, we shall denote Caputo type fractional q-derivative by the symbol ${}_*D_q^{\alpha}$.

For $\alpha\in\Re^+$ and $a<x$, we have [16]

$$I_{q,a}^{\alpha}\,{}_*D_{q,a}^{\alpha}f\left(x\right)=f\left(x\right)-\sum_{k=0}^{n-1}\frac{D_q^kf\left(a\right)}{\left[k\right]_q!}\left(x-a\right)_q^k,\ n-1<\alpha\le n,\ n\in\mathbb{N}. \tag{1.19}$$

q-differential transform [8]

The q-differential transform $F_q\left(k\right)$ of the function $f\left(x\right)$ at $x=a$ is defined as follows

$$F_q\left(k\right)=\frac{1}{\left[k\right]_q!}\left[D_q^kf\left(x\right)\right]_{x=a}. \tag{1.20}$$

The inverse q-differential transform of $F_q(k)$ is given by:

$$f(x)=\sum_{k=0}^{\infty}F_q(k)(x-a)_q^k. \tag{1.21}$$

Here $f(x)$ is called original function and $F_q(k)$ transformed function.

3. Generalized q-Taylor's formula involving Caputo fractional q-derivatives

Our main result of this section is generalized q-Taylor's formula involving Caputo fractional q-derivatives given as Theorem 2. We also establish as Theorem 1, a generalized mean value theorem for Caputo fractional q-derivative and a lemma which is required to prove main result.

Theorem1. If $f(x), {}_*D_{q,a}^{\alpha}f(x)$ are continuous functions for $x\in[a,b]$ and $0<\alpha\le 1$, then $\exists c\in(a,b)$ and $\hat{q}\in(0,1)$ such that

$$f(x)=f(a)+\frac{1}{\Gamma_q(\alpha+1)}\,{}_*D_{q,a}^{\alpha}f(c)(x-a)_q^{\alpha} \qquad \forall q\in(\hat{q},1), \tag{2.1}$$

Proof. On applying Riemann-Liouville fractional q-integral operator $I_{q,a}^{\alpha}$ given by (1.12), to function ${}_*D_{q,a}^{\alpha}f(x)$,we get

$$I_{q,a}^{\alpha}\,{}_*D_{q,a}^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha)}\int_a^x(x-tq)_q^{\alpha-1}\,{}_*D_{q,a}^{\alpha}f(t)\,d_qt. \tag{2.2}$$

Using second mean value theorem for q-integral given by (1.6), for $c\in(a,x)$ we get

$$I_{q,a}^{\alpha}\,{}_*D_{q,a}^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha)}\,{}_*D_{q,a}^{\alpha}f(c)\int_a^x(x-tq)_q^{\alpha-1}d_qt. \tag{2.3}$$

Using (1.12) and (1.16) (with $\lambda=0$), it is simplified to

$$I_{q,a}^{\alpha}\,{}_*D_{q,a}^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha+1)}\,{}_*D_{q,a}^{\alpha}f(c)(x-a)_q^{\alpha}. \tag{2.4}$$

Using (1.19) for $n=1$, we get (2.1).

In case $\alpha=1$, the generalized mean value theorem for Caputo fractional q-derivative (2.1) reduces to q-mean value theorem [10].

For $q\to 1$, Theorem 1 gives the generalized mean value theorem for Caputo derivative given by Odibat et al. [13].

Lemma. If $\left({}_*D_{q,a}^{\alpha}\right)^i f(x),\left({}_*D_{q,a}^{\alpha}\right)^{i+1} f(x)$ are continuous functions on [a, b], then for $0<\alpha\le 1$ we have

$$I_{q,a}^{i\alpha}\left({}_*D_{q,a}^{\alpha}\right)^i f(x)-I_{q,a}^{(i+1)\alpha}\left({}_*D_{q,a}^{\alpha}\right)^{i+1} f(x)=\frac{\left({}_*D_{q,a}^{\alpha}\right)^i f(a)}{\Gamma_q(i\alpha+1)}(x-a)_q^{i\alpha}. \tag{3.1}$$

where $\left({}_*D_{q,a}^{\alpha}\right)^i={}_*D_{q,a}^{\alpha}\cdot{}_*D_{q,a}^{\alpha}\cdots{}_*D_{q,a}^{\alpha}$ (i times).

Proof. The left side of (3.1), using (1.15) can be written as

$$I_{q,a}^{i\alpha}\left[\left({}_*D_{q,a}^{\alpha}\right)^i f(x)-I_{q,a}^{\alpha}\left({}_*D_{q,a}^{\alpha}\right)^{i+1} f(x)\right] \tag{3.2}$$

$$=I_{q,a}^{i\alpha}\left[\left({}_*D_{q,a}^{\alpha}\right)^i f(x)-\left(I_{q,a}^{\alpha}\,{}_*D_{q,a}^{\alpha}\right)\left({}_*D_{q,a}^{\alpha}\right)^i f(x)\right]. \tag{3.3}$$

Using (1.19) (with n=1), it becomes

$$I_{q,a}^{i\alpha}\left[\left({}_*D_{q,a}^{\alpha}\right)^i f(a)\right], \tag{3.4}$$

Further, using result (1.16), we get right side of (3.1).

Theorem 2. If $\left({}_*D_{q,a}^{\alpha}\right)^k f(x)$ for all $k=0,1,\ldots,n+1, n\in N$ are continuous functions on [a, b] and $0<\alpha\le 1$, then $\exists c\in(a,b)$ and $\hat{q}\in(0,1)$ such that

$$f(x)=\sum_{i=0}^{n}\frac{\left({}_*D_{q,a}^{\alpha}\right)^i f(a)}{\Gamma_q(i\alpha+1)}(x-a)_q^{i\alpha}+\frac{\left({}_*D_{q,a}^{\alpha}\right)^{n+1} f(c)}{\Gamma_q((n+1)\alpha+1)}(x-a)_q^{(n+1)\alpha} \quad \forall q\in(\hat{q},1). \tag{3.5}$$

where $\left({}_*D_{q,a}^{\alpha}\right)^i = {}_*D_{q,a}^{\alpha}\cdot{}_*D_{q,a}^{\alpha}\ldots{}_*D_{q,a}^{\alpha}$ (i times).

Proof. From (3.1), we have

$$\sum_{i=0}^{n}\left\{I_{q,a}^{i\alpha}\left({}_*D_{q,a}^{\alpha}\right)^i f(x)-I_{q,a}^{(i+1)\alpha}\left({}_*D_{q,a}^{\alpha}\right)^{i+1} f(x)\right\}=\sum_{i=0}^{n}\frac{(x-a)_q^{i\alpha}}{\Gamma_q(i\alpha+1)}\left({}_*D_{q,a}^{\alpha}\right)^i f(a). \tag{3.6}$$

On simplification, it becomes

$$f(x)-I_{q,a}^{(n+1)\alpha}\left({}_*D_{q,a}^{\alpha}\right)^{n+1} f(x)=\sum_{i=0}^{n}\frac{\left({}_*D_{q,a}^{\alpha}\right)^i f(a)}{\Gamma_q(i\alpha+1)}(x-a)_q^{i\alpha}. \tag{3.7}$$

Using definition (1.12), we get

$$f(x)=\sum_{i=0}^{n}\frac{\left({}_*D_{q,a}^{\alpha}\right)^i f(a)}{\Gamma_q(i\alpha+1)}(x-a)_q^{i\alpha}+\frac{1}{\Gamma_q((n+1)\alpha)}\int_a^x(x-tq)_q^{(n+1)\alpha-1}\left({}_*D_{q,a}^{\alpha}\right)^{n+1} f(t)\,d_qt. \tag{3.8}$$

Using second mean value theorem for q-integral given by (1.6), for $c\in(a,x)$ we get

$$f(x)=\sum_{i=0}^{n}\frac{(x-a)_q^{i\alpha}}{\Gamma_q(i\alpha+1)}\left({}_*D_{q,a}^{\alpha}\right)^i f(a)+\frac{\left({}_*D_{q,a}^{\alpha}\right)^{n+1} f(c)}{\Gamma_q((n+1)\alpha)}\int_a^x(x-tq)_q^{(n+1)\alpha-1}\,d_qt. \tag{3.9}$$

Using (1.12) and (1.16) (with $\lambda=0$), in the integral in (3.9), we get result (3.5).

The radius of convergence, R for the generalized q-Taylor's series

$$f(x)=\sum_{i=0}^{\infty}\frac{\left({}_*D_{q,a}^{\alpha}\right)^i f(a)}{\Gamma_q(i\alpha+1)}(x-a)_q^{i\alpha}, \tag{3.10}$$

is given by

$$R=\lim_{n\to\infty}\left|\frac{\Gamma_q(n\alpha+1)}{\Gamma_q((n+1)\alpha+1)}\frac{\left({}_*D_{q,a}^{\alpha}\right)^{n+1} f(a)}{\left({}_*D_{q,a}^{\alpha}\right)^{n} f(a)}\right|. \tag{3.11}$$

In case $\alpha=1$, the Caputo generalized q-Taylor's formula (3.5) reduces to q-Taylor's formula given by Jackson [2]. For $q\to 1$, Caputo generalized q-Taylor's formula (3.5) gives the generalized Taylor's formula involving Caputo derivatives given by Odibat et al. [13].

4. GENERALIZED *q*-DIFFERENTIAL TRANSFORM

We define the generalized q-differential transform $F_{q,\alpha}(k)$ of function $f(x)$ at point $x=a$ as follows:

$$F_{q,\alpha}(k)=\frac{1}{\Gamma_q(\alpha k+1)}\left[\left({}_*D_{q,a}^{\alpha}\right)^k f(x)\right]_{x=a}, \tag{4.1}$$

where $0<\alpha\le 1$ and $\left({}_*D_{q,a}^{\alpha}\right)^k = {}_*D_{q,a}^{\alpha}\cdot{}_*D_{q,a}^{\alpha}\cdots{}_*D_{q,a}^{\alpha}$ (k times).

The inverse generalized q-differential transform of $F_{q,\alpha}(k)$ in view of (3.10), is given by:

$$f(x)=\sum_{k=0}^{\infty}F_{q,\alpha}(k)(x-a)_q^{k\alpha}. \tag{4.2}$$

For $\alpha=1$, (4.1) and (4.2) yield the q-differential transform and its inverse given by (1.20) and (1.21) respectively. For $q\to 1$, (4.1) and (4.2) give the generalized differential transform and its inverse defined by Odibat et al. [15].

Some basic properties of the generalized *q*-differential transform are as given below:

Theorem3. If $F_{q,\alpha}(k), U_{q,\alpha}(k)$ and $V_{q,\alpha}(k)$ are generalized q-differential transforms of functions $f(x), u(x)$ and $v(x)$ at point $x=a$ respectively, then the following results hold.

(a). If $f(x)=u(x)\pm v(x)$, then $F_{q,\alpha}(k)=U_{q,\alpha}(k)\pm V_{q,\alpha}(k)$.

(b). If $f(x)=cu(x)$, c is constant, then $F_{q,\alpha}(k)=cU_{q,\alpha}(k)$.

(c). If $f(x)=(x-a)_q^{n\alpha}, n\in \boldsymbol{N}$, then $F_{q,\alpha}(k)=\delta(k-n)$, where

$$\delta(k)=\begin{cases}1, & \text{when } k=0\\ 0, & \text{otherwise}\end{cases}.$$

(d). If $f(x)={}_*D_{q,a}^{\alpha}u(x)$, then $F_{q,\alpha}(k)=\dfrac{\Gamma_q(\alpha(k+1)+1)}{\Gamma_q(\alpha k+1)}U_{q,\alpha}(k+1)$.

Proof. By linearity of generalized q-differential transform, we can easily obtain results (a) and (b).

(c). Writing

$$f(x)=\sum_{k=0}^{\infty}\delta(k-n)(x-a)_q^{k\alpha}. \tag{4.3}$$

We get in view of (4.2) $F_{q,\alpha}(k)=\delta(k-n)$.

(d). Taking $f(x)={}_*D_{q,a}^{\alpha}u(x)$ in (4.1), we get

$$\begin{aligned}F_{q,\alpha}(k)&=\frac{1}{\Gamma_q(\alpha k+1)}\left[\left({}_*D_{q,a}^{\alpha}\right)^k {}_*D_{q,a}^{\alpha}u(x)\right]_{x=a}\\ &=\frac{1}{\Gamma_q(\alpha k+1)}\left[\left({}_*D_{q,a}^{\alpha}\right)^{k+1} u(x)\right]_{x=a}\\ &=\frac{\Gamma_q(\alpha(k+1)+1)}{\Gamma_q(\alpha k+1)}U_{q,\alpha}(k+1).\end{aligned}$$

Theorem 4. If $F_{q,\alpha}(k), U_{q,\alpha}(k)$ and $V_{q,\alpha}(k)$ are generalized q-differential transforms of functions $f(x), u(x)$ and $v(x)$ at the point $x=0$ respectively, then the following results hold.

(a). If $f(x)=u(cx)$, c is constant, then

$$F_{q,\alpha}(k)=c^{k\alpha}U_{q,\alpha}(k). \tag{4.4}$$

(b). If $f(x)=u(x)v(x)$, then

$$F_{q,\alpha}(k)=\sum_{i=0}^{k}U_{q,\alpha}(k-i)V_{q,\alpha}(i). \tag{4.5}$$

Proof.

(a). On taking $a=0$ and replacing x by cx in inverse generalized q-differential transform (4.2) for $u(x)$, we get

$$u(cx)=\sum_{k=0}^{\infty}U_{q,\alpha}(k)(cx)^{k\alpha}=\sum_{k=0}^{\infty}U_{q,\alpha}(k)c^{k\alpha}x^{k\alpha}, \tag{4.6}$$

which again in view of (4.2) gives the result (4.4).

(b). Using definition of inverse generalized q-differential transform (4.2) at $a=0$ for the functions $f(x)$, $u(x)$ and $v(x)$, we get

$$\sum_{k=0}^{\infty}F_{q,\alpha}(k)x^{k\alpha}=\sum_{k=0}^{\infty}U_{q,\alpha}(k)x^{k\alpha}\sum_{i=0}^{\infty}V_{q,\alpha}(i)x^{i\alpha}. \tag{4.7}$$

On using series manipulation and then equating the coefficients of $x^{k\alpha}$, we arrive at (4.5).

5. SOLUTIONS OF FRACTIONAL *q*-DIFFERENCE EQUATIONS

In this section, we shall solve four fractional q-difference equations involving Caputo fractional q-derivative by applying generalized q-differential transform and its inverse given by (4.1) and (4.2) respectively.

Problem 1. For $0<|q|<1, 0<\alpha\le 1$ and $\alpha\in\Re$, solution of nonlinear Caputo fractional q-Riccati type problem

$${}_{*}D_q^{\alpha}y(x)=y(x)y(qx)+a\Gamma_q(\alpha+1)-a^2q^{\alpha}x^{2\alpha}, x>0 \tag{5.1}$$

with initial condition $y(0)=0$, is given by

$$y(x)=ax^{\alpha}. \tag{5.2}$$

Taking generalized q-differential transform (4.1) of (5.1) and initial condition (5.2), using Theorems 3, 4, 5 and 6, we get

$$\frac{\Gamma_q(\alpha(k+1)+1)}{\Gamma_q(\alpha k+1)}Y_{q,\alpha}(k+1)=\sum_{i=0}^{k}q^{\alpha i}Y_{q,\alpha}(i)Y_{q,\alpha}(k-i)+a\Gamma_q(\alpha+1)\delta(k)-a^2q^{\alpha}\delta(k-2). \tag{5.3}$$

and

$$Y_{q,\alpha}(0)=0. \tag{5.4}$$

Utilizing recurrence relation (5.3) and transformed initial condition (5.4), we obtain

$$\begin{aligned}&Y_{q,\alpha}(1)=a,\\&Y_{q,\alpha}(2)=0,\\&Y_{q,\alpha}(3)=0,\\&.\\&.\\&.\end{aligned} \tag{5.5}$$

and all other terms vanish.

Using these values of $Y_{q,\alpha}(k)$ in equation (4.2), we obtain the exact solution (5.2).

This may be verified by direct substitution in the equation (5.1).

Problem 2. For $0<|q|<1, 0<\alpha\le 1$ and $\lambda, a\in\Re$, solution of Caputo fractional q-difference equation

$${}_*D_q^{\alpha} y(x) = \lambda x^{\alpha} y\left(q^{-\alpha} x\right), x > 0, \tag{5.6}$$

with initial condition

$$y(0) = a, \tag{5.7}$$

is given by

$$y(x) = a \, {}_qE_{\alpha,2,1}(\lambda, x), \tag{5.8}$$

where ${}_qE_{\alpha,2,1}(\lambda, x)$ stands for generalized q-Mittag-Leffler function given by (1.9).

At the point $x = 0$, (5.6) gives derived condition ${}_*D_q^{\alpha} y(0) = 0$ (5.9)

On applying generalized q-differential transform (4.1) to equation (5.6) and using Theorems 4, 5 and 6, it transforms to

$$\frac{\Gamma_q(\alpha(k+1)+1)}{\Gamma_q(\alpha k+1)} Y_{q,\alpha}(k+1) = \lambda \sum_{i=0}^{k} \delta(i-1) q^{-\alpha^2 (k-i)} Y_{q,\alpha}(k-i) \tag{5.10}$$

which gives following recurrence relation

$$\frac{\Gamma_q(\alpha(k+1)+1)}{\Gamma_q(\alpha k+1)} Y_{q,\alpha}(k+1) = \lambda q^{-\alpha^2 (k-1)} Y_{q,\alpha}(k-1), k \geq 1 \tag{5.11}$$

The generalized q-differential transform of initial conditions (5.7) and (5.9) are given by

$$Y_{q,\alpha}(0) = a \text{ and } Y_{q,\alpha}(1) = 0 \tag{5.12}$$

Utilizing recurrence relation (5.11) and transformed initial conditions (5.12), we obtain

$$Y_{q,\alpha}(2) = \lambda a \frac{\Gamma_q(\alpha+1)}{\Gamma_q(2\alpha+1)},$$

$$Y_{q,\alpha}(4) = \lambda^2 a \frac{\Gamma_q(3\alpha+1)}{\Gamma_q(4\alpha+1)} \frac{\Gamma_q(\alpha+1)}{\Gamma_q(2\alpha+1)} q^{-2\alpha^2},$$

$$Y_{q,\alpha}(6) = \lambda^3 a \frac{\Gamma_q(5\alpha+1)}{\Gamma_q(6\alpha+1)} \frac{\Gamma_q(3\alpha+1)}{\Gamma_q(4\alpha+1)} \frac{\Gamma_q(\alpha+1)}{\Gamma_q(2\alpha+1)} q^{-4\alpha^2}, \tag{5.13}$$

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and $Y_{q,\alpha}(1) = Y_{q,\alpha}(3) = Y_{q,\alpha}(5) = \ldots = 0$.

Using values of $Y_{q,\alpha}(k)$ in equation (4.2), we obtain the exact solution (5.8).

The above problem is a particular case of the problem considered by Abdeljawad [17], which is solved by successive approximation method.

Problem 3. For $0 <| q |< 1$ and $\lambda, a, b, c \in \Re$, the solution of Caputo fractional q-difference equation

$${}_*D_{q,a}^{3/2} y(x) = \lambda y(x), x > a, \tag{5.14}$$

with initial conditions $y(a) = b$ and $D_q y(a) = c$, is given by

$$y(x) = b \, {}_qE_{3/2}(\lambda, x-a) + c(x-a) \, {}_qE_{3/2,2}(\lambda, x-aq), \tag{5.15}$$

where ${}_qE_\alpha\left(\lambda, x-a\right)$ and ${}_qE_{\alpha,\beta}\left(\lambda, x-a\right)$ are q-Mittag-Leffler functions given by (1.7) and (1.8) respectively.
On applying generalized q-differential transform (4.1) with $\alpha = 1/2$ to equation (5.14), using Theorems 3(b) and 5, it transforms to

$$\frac{\Gamma_q\left(\frac{1}{2}(k+3)+1\right)}{\Gamma_q\left(k/2+1\right)} Y_{q,1/2}\left(k+3\right) = \lambda Y_{q,1/2}\left(k\right), \tag{5.16}$$

where $Y_{q,\alpha}\left(k\right)$ denotes the generalized q-differential transform of function $y\left(x\right)$.
Taking generalized q-differential transforms of given initial conditions, we get

$$Y_{q,1/2}\left(0\right) = b, Y_{q,1/2}\left(1\right) = 0, Y_{q,1/2}\left(2\right) = c. \tag{5.17}$$

Utilizing recurrence relation (5.16) and transformed initial conditions (5.17), we obtain

$$\begin{aligned} &Y_{q,1/2}\left(3\right) = \frac{1}{\Gamma_q\left(5/2\right)}\lambda b, Y_{q,1/2}\left(6\right) = \frac{1}{\Gamma_q\left(4\right)}\lambda^2 b, Y_{q,1/2}\left(9\right) = \frac{1}{\Gamma_q\left(11/2\right)}\lambda^3 b, \ldots \\ &Y_{q,1/2}\left(4\right) = 0, Y_{q,1/2}\left(7\right) = 0, Y_{q,1/2}\left(10\right) = 0, \ldots.. \\ &Y_{q,1/2}\left(5\right) = \frac{1}{\Gamma_q\left(7/2\right)}\lambda c, Y_{q,1/2}\left(8\right) = \frac{1}{\Gamma_q\left(5\right)}\lambda^2 c, Y_{q,1/2}\left(11\right) = \frac{1}{\Gamma_q\left(13/2\right)}\lambda^3 c \ldots. \end{aligned} \tag{5.18}$$

and so on.
Using values of $Y_{q,1/2}\left(k\right)$ in equation (4.2), we obtain

$$\begin{aligned} y\left(x\right) &= b + \frac{1}{\Gamma_q\left(5/2\right)}\lambda b(x-a)_q^{3/2} + \frac{1}{\Gamma_q\left(4\right)}\lambda^2 b(x-a)_q^{3} + \frac{1}{\Gamma_q\left(11/2\right)}\lambda^3 b(x-a)_q^{9/2} + \ldots \\ &+ c(x-a) + \frac{1}{\Gamma_q\left(7/2\right)}\lambda c(x-a)_q^{5/2} + \frac{1}{\Gamma_q\left(5\right)}\lambda^2 c(x-a)_q^{4} + \frac{1}{\Gamma_q\left(13/2\right)}\lambda^3 c(x-a)_q^{11/2} + \ldots., \end{aligned} \tag{5.19}$$

which in view of definitions (1.8) and (1.7) of q-Mittag-Leffler functions and (1.4), gives the exact solution (5.15). Taking $q \to 1$, above problem reduces to the problem considered by Kilbas [12], which is solved by Laplace transform method.

Problem 4. For $0 < |q| < 1$ and $\lambda, a, b, c \in \Re$, the solution of fractional q-difference equation

$${}_*D_{q,a}^{5/4} y\left(x\right) - \lambda\, {}_*D_{q,a}^{1/2} y\left(x\right) = 0, x > a, \tag{5.20}$$

with initial conditions $y\left(a\right) = b$ and $\mathrm{D}_q y\left(a\right) = c$, is given by

$$y\left(x\right) = b + c\left(x-a\right) {}_qE_{3/4,2}\left(\lambda, x-aq\right). \tag{5.21}$$

Taking generalized q-differential transform (4.1) with $\alpha = 1/4$ of (5.20) and using Theorems 3(b) and 5, it transforms to

$$\frac{\Gamma_q\left(\frac{1}{4}(k+5)+1\right)}{\Gamma_q\left(k/4+1\right)} Y_{q,1/4}\left(k+5\right) - \lambda \frac{\Gamma_q\left(\frac{1}{4}(k+2)+1\right)}{\Gamma_q\left(k/4+1\right)} Y_{q,1/4}\left(k+2\right) = 0. \tag{5.22}$$

Taking generalized q-differential transforms of given initial conditions, we get

$$Y_{q,1/4}\left(0\right) = b, Y_{q,1/4}\left(1\right) = 0, Y_{q,1/4}\left(2\right) = 0, Y_{q,1/4}\left(3\right) = 0, Y_{q,1/4}\left(4\right) = c. \tag{5.23}$$

Utilizing recurrence relation (5.22) and transformed initial conditions (5.23), we obtain

$$\begin{aligned} &Y_{q,1/4}\left(7\right) = \frac{1}{\Gamma_q\left(11/4\right)}\lambda c, Y_{q,1/4}\left(10\right) = \frac{1}{\Gamma_q\left(7/2\right)}\lambda^2 c, Y_{q,1/4}\left(13\right) = \frac{1}{\Gamma_q\left(17/4\right)}\lambda^3 c \ldots. \\ &Y_{q,1/4}\left(5\right) = 0, Y_{q,1/4}\left(6\right) = 0, Y_{q,1/4}\left(8\right) = 0, Y_{q,1/4}\left(9\right) = 0, \ldots.. \end{aligned} \tag{5.24}$$

and so on.

Using these values of $Y_{q,1/4}(k)$ in equation (4.2), we obtain

$$y(x)=b+c(x-a)+\frac{1}{\Gamma_q\left(11/4\right)}\lambda c(x-a)_q^{7/4}+\frac{1}{\Gamma_q\left(7/2\right)}\lambda^2 c(x-a)_q^{10/4}+\frac{1}{\Gamma_q\left(17/4\right)}\lambda^3 c(x-a)_q^{13/4}+....,\tag{5.25}$$

which in view of (1.8) and (1.4) gives the exact solution (5.21).

Taking $q \to 1$, above problem reduces to the problem considered by Kilbas [12], which is solved by Laplace transform method.

REFERENCES

1. Jackson, F.H.: On *q*-functions and a certain difference operator. Transactions of the Royal Society of Edinburgh **46**, 253-281 (1908).
2. Jackson, F.H.: A *q*-form of Taylor's theorem. Messenger of Mathematics **38**, 62-64 (1909).
3. Jackson, F.H.: On *q*-difference equations. American Journal of Mathematics **32**, 305-314 (1910).
4. Al-Salam, W.A.: Some fractional *q*-integrals and *q*-derivatives. Proceedings of the Edinburgh Mathematical Society **15,** 135-140 (1966).
5. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent. Part II. Geophysical Journal of the Royal Astronomical Society **13**, 529–539 (1967).
6. Agarwal, R.P.: Certain fractional *q*-integrals and *q*-derivatives. Proceedings of the *Cambridge* Philosophical Society, **66**, 365-370 (1969).
7. Zhou J.K.: Differential Transformation and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, China, 1986.
8. Jing, S.C., Fan, H.Y.: *q*-Taylor's formula with its *q*-remainder, Communications in Theoretical Physics **23** (1), 117–120 (1995).
9. Frenkel, E.: Deformations of the KdV Hierarchy and related soliton equations, International Mathematics Research Notices **2**, 55-76 (1996).
10. Rajković, P.M., Stanković, M.S., Marinković, S.D.: Mean value theorems in q-calculus. Matematicki Vesnik **54** (3-4), 171-178 (2002).
11. Gasper, G. and Rahman, M.: Basic Hypergeometric Series, 2nd ed, Encyclopedia of Mathematics and its Applications 96, Cambridge University Press, Cambridge, 2004.
12. Kilbas, A. A., Srivastava and H. M., Trujiilo, J. J.: Theory and Applications of Fractional Differential Equations, Amsterdam, Netherlands, Elsevier, 2006.
13. Odibat, Z., Shawagfeh, N.: Generalized Taylor's formula, Applied Mathematics and Computation **186**, 286–293 (2007).
14. Rajković, P. M., Marinković, S. D., Stanković, M. S.: Fractional Integrals and Derivatives in *q*-calculus, Applicable Analysis and Discrete Mathematics **1**, 311-323, (2007).
15. Odibat, Z., Momani, S., Erturk, V.S.: Generalized differential transform method: Application to differential equations of fractional order. Applied Mathematics and Computation **197**, 467–477(2008).
16. Abdeljawad, T., Baleanu, D.: Caputo *q*-fractional initial value problems and a q-analogue Mittag–Leffler function. Communications in Nonlinear Science and Numerical Simulation **16(**12), 4682-4688 (2011).
17. Abdeljawad, T., Benli B.: A Quantum Generalized Mittag-Leffler Function Via Caputo *q*-Fractional Equations, arXiv:1102.1585v1 [math.DS] 8 Feb 2011.