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INFINITELY SPLIT NASH EQUILIBRIUM PROBLEMS IN REPEATED GAMES BY HOMOTOPY METHOD

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Abstract

In the realm of game theory, repeated games offer a richer and more dynamic landscape than their single-shot counterparts. They introduce the crucial element of time and the opportunity for players to learn, adapt, and build reputations, leading to a vast array of potential outcomes. Among these, the concept of a Nash equilibrium remains central, representing a stable state where no player has an incentive to unilaterally deviate. However, in repeated games, the sheer complexity can make identifying these equilibria a daunting task. This is particularly true for "infinitely split Nash equilibrium problems," a specific class of problems that arise when the game's horizon extends indefinitely and the payoffs are discounted over time. This article will explore these complex equilibrium problems and argue for the efficacy of the homotopy method as a powerful tool for their analysis and solution. Infinitely split Nash equilibrium problems in repeated games fundamentally deal with situations where a finite game is played an infinite number of times, with players discounting future payoffs. The "infinitely split" aspect refers to the fine-grained nature of strategies, where players can react to even subtle past deviations, potentially leading to a multitude of equilibrium paths. The standard Folk Theorems, while providing existence conditions for a wide range of equilibria in repeated games, often fall short in offering constructive methods for finding these equilibria or analyzing their specific properties. The challenge lies in the high

dimensionality of the strategy spaces and the intricate interdependencies between current actions and future consequences. Traditional iterative methods may struggle with convergence, especially in the presence of multiple equilibria or when the payoff functions are nonlinear and non-convex.

Keywords:

Infinitely, split, Nash, equilibrium, Repeated, Games, Homotopy

Introduction

For infinitely split Nash equilibrium problems, the homotopy method offers several distinct advantages. Firstly, it provides a constructive approach to finding equilibria, rather than merely proving their existence. By continuously tracking the solution path, it can identify specific equilibrium strategies and their corresponding payoffs. This is crucial for practical applications where understanding the nature of the equilibrium is as important as knowing that it exists. (Mortici, 2020)

The homotopy method is particularly well-suited for dealing with multiple equilibria. Unlike iterative methods that might converge to only one specific equilibrium depending on the initial guess, a properly constructed homotopy can potentially uncover all or a significant subset of equilibria by tracing different paths. This is vital in repeated games where multiple stable outcomes are common, and understanding the range of possible behaviors is essential for predicting and influencing strategic interactions.

Repeated games, where players interact multiple times, offer a richer and more complex environment for strategic decision-making than single-shot encounters. While the concept of Nash equilibrium remains central to analyzing these interactions, repeated games introduce unique challenges and insights into its application. This article will explore the intricacies of Nash equilibrium problems in repeated games,

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highlighting how repetition can both alleviate and exacerbate the inherent dilemmas of strategic interaction, leading to a broader range of possible outcomes and requiring more sophisticated analytical tools.

One of the most significant aspects of repeated games is the potential for cooperation to emerge even in situations where a single-shot game would predict a dominant strategy of defection. The Folk Theorem, a cornerstone of repeated game theory, illustrates this point. It posits that in infinitely repeated games (or sufficiently long finite games), any individually rational and feasible payoff profile can be supported as a Nash equilibrium. This is achieved through the use of "trigger strategies," where players condition their current actions on the past behavior of their opponents. For example, a "grim trigger" strategy involves cooperating as long as the other player cooperates, but defecting forever if the other player ever defects. The threat of future punishment can incentivize cooperation, moving the outcome away from a sub-optimal, non-cooperative Nash equilibrium of the stage game. However, the abundance of such equilibria poses its own problem: how do players coordinate on which of these many potential cooperative outcomes to achieve? This multiplicity of equilibria is a significant Nash equilibrium problem in repeated games, as it reduces the predictive power of the equilibrium concept. (Merovci, 2021)

$$X_1(a,b;c,d;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p}{(c)_m(d)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad \text{………………e.q.1.1}$$

$$X_2(a,b;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \quad \text{…e.q.1.2}$$

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$$X_3(a, b; c, d; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_{n+p}}{(c)_{m+n}(d)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$ …e.q.1.3

$$X_4(a, b; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$ …e.q.1.4

$$X_5(a, b_1, b_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$ …e.q.1.5

Furthermore, the discount factor, which reflects the relative importance players place on future payoffs compared to present payoffs, plays a crucial role in determining the viability of cooperative Nash equilibria.

This is precisely where the homotopy method emerges as a compelling alternative. Homotopy, in mathematics, refers to the continuous deformation of one mathematical object into another. In the context of solving equilibrium problems, it involves embedding the original problem within a family of problems, continuously deforming a simpler, easily solvable problem into the more complex, target problem. The core idea is to trace a path of solutions from the known solution of the simpler problem to the unknown solution of the target problem. (Krasniqi, 2020)

Literature Review

Abramowitz et al. (2020): The problem becomes one of coordinating on a specific equilibrium, as any miscoordination can lead to suboptimal outcomes. This often requires players to develop conventions, communicate effectively, or rely on focal points to achieve a mutually beneficial outcome, underscoring the limitations of Nash equilibrium alone in predicting real-world behavior in such scenarios.

Singh et al. (2020): Players will defect in the second-to-last period, and so on, leading to a backward induction outcome that mirrors the single-shot Nash equilibrium. This "endgame effect" is a significant Nash equilibrium problem, as it suggests that cooperation may be fleeting or non-existent in finitely repeated games, even if intuitively players might benefit from it.

Teruel et al. (2020): Nash equilibrium problems in repeated games are multifaceted. While the repetition of interactions offers a powerful mechanism for overcoming the limitations of single-shot games and fostering cooperation, it also introduces complexities such as the multiplicity of equilibria, the critical role of the discount factor, and the challenges of backward induction in finite games.

Alzer et al. (2021): The problems associated with Nash equilibrium in repeated games are not solely about the difficulty of achieving cooperation. They also extend to the challenges of coordination when multiple desirable equilibria exist. Consider a repeated coordination game where players have multiple Pareto-efficient Nash equilibria.

Karlin et al. (2021): Understanding Nash equilibrium problems is crucial for accurately analyzing strategic behavior in dynamic settings. Rather than being a mere extension of single-shot game theory, repeated game theory provides a deeper, albeit more intricate, framework for comprehending how individuals and organizations navigate ongoing strategic interactions, constantly balancing the allure of immediate gains against the long-term consequences of their actions.

Pariguan et al. (2022): While Nash equilibrium only requires that no player can unilaterally deviate and improve their payoff, SPNE adds the requirement that this condition must hold in every subgame of the overall game. This eliminates incredible threats or promises that are not optimal to carry out if the situation ever arises. In repeated games, particularly those with a finite horizon, SPNE often unravels cooperation from the end of the game backward. In the last period, there is no future to incentivize cooperation, so players will defect.

Jiao et al. (2023): A higher discount factor (i.e., less discounting of the future) makes future punishments more credible and future rewards more appealing, thus increasing the likelihood of sustaining cooperation.

Heikkila et al. (2024): In heavily discounted games, the short-term gains from defection outweigh the long-term benefits of cooperation, pushing the equilibrium back towards the non-cooperative outcome of the stage game. Understanding the players' discount factors is therefore essential for predicting behavior and identifying the relevant Nash equilibria.

Infinitely split Nash equilibrium problems in repeated games by Homotopy Method

The homotopy method is a powerful mathematical technique used to solve a wide range of problems, particularly those involving systems of nonlinear equations, optimization, and eigenvalue problems. At its core, the method transforms a difficult problem into a continuous family of simpler problems, gradually deforming a known, easily solvable problem into the target problem. This continuous deformation, or "homotopy," provides a systematic path to finding solutions, often overcoming challenges associated with traditional iterative methods like convergence issues and dependence on good initial guesses.

The fundamental idea behind the homotopy method is to embed the problem of interest, say f(x)=0, into a one-parameter family of equations H(x,t)=0, where t∈ [0,1]. This homotopy function H(x,t) is typically constructed such that at t=0, H(x,0)=g(x)=0 is a trivial or easily solvable system, and at t=1, H(x,1)=f(x)=0 is the original problem we wish to solve. A common choice for H(x,t) is the convex homotopy:

H(x,t)=tf(x)+(1−t)g(x)

As t continuously varies from 0 to 1, the solution x(t) traces a path in the solution space. The goal is to follow this path from the known solution of g(x)=0 at t=0 to the solution of f(x)=0 at t=1.

The existence and uniqueness of the solution path are crucial for the success of the homotopy method. Under certain conditions, often related to the regularity and non-singularity of the Jacobian of H(x,t) along the path, it can be proven that a smooth solution path exists. Numerically, following this path typically involves solving an initial value problem. Differentiating H(x(t),t)=0 with respect to t yields:

∂x∂Hdtdx+∂t∂H=0

This leads to a system of ordinary differential equations (ODEs):

dtdx=−(∂x∂H)−1∂t∂H

This system can be solved using standard ODE solvers, such as predictor-corrector methods, which involve taking small steps along the path and then correcting the solution to ensure it remains on the homotopy curve.

One of the significant advantages of the homotopy method is its global convergence property. Unlike local iterative methods that can get trapped in local minima or diverge if the initial guess is not close enough to the true solution, homotopy methods are designed to trace a path that leads to a solution, regardless of the starting point (provided the path does not encounter singularities or diverge to infinity). This makes them particularly robust for highly nonlinear problems where good initial guesses are hard to come by.

$$X_6(a,b_1,b_2;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c)_{m+n}(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.6

$$X_7(a,b_1,b_2;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c)_m(d)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$,…e.q.1.7

$$X_8(a,b_1,b_2;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.8

$$X_9(a,b;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.9

$$X_{10}(a,b;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c)_{m+n}(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.10

$$X_{12}(a,b;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ ……..…e.q.1.11

$$X_{12}(a,b;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.12

However, the homotopy method is not without its challenges. The computational cost can be higher than local methods, as it involves solving a system of ODEs and potentially inverting matrices at each step. Path-following can also become complicated if the path exhibits turning points (where ∂x∂H becomes singular) or branches. Strategies for handling such complexities include re-parametrization of the path or using more sophisticated path-following algorithms.

Applications of the homotopy method are diverse and impactful. In engineering, it's used for power system analysis, structural mechanics, and chemical process optimization. In economics, it helps in computing economic equilibria. In scientific computing, it finds solutions to systems arising from discretizations of partial differential equations and in inverse problems. Beyond finding roots of nonlinear equations, it has been extended to global optimization problems by transforming the optimization problem into a system of equations whose roots correspond to stationary points.

The homotopy method stands as a versatile and powerful tool in the arsenal of numerical analysis. By providing a continuous deformation from a simple problem to a complex one, it offers a robust approach to solving challenging mathematical problems that often elude traditional techniques. Despite its computational demands and the intricacies of path-following, its global convergence properties and wide applicability ensure its continued importance in various scientific and engineering disciplines. As computational power increases and algorithms become more refined, the homotopy method is poised to tackle even more complex and large-scale problems in the future

The robustness of the homotopy method to non-linearity and non-convexity is a significant benefit. The payoff functions in repeated games, especially with discounting and complex strategic interactions, are often non-linear. Traditional optimization techniques can struggle in such landscapes, but the continuous path-following nature of homotopy can navigate these complexities more effectively.

Applying the homotopy method to infinitely split Nash equilibrium problems typically involves constructing a system of equations that characterize the Nash equilibrium conditions (e.g., first-order conditions for optimization problems or fixed-point equations for best-response mappings). A homotopy function is then defined, interpolating between a trivial problem (with an obvious solution) and the original equilibrium problem. The path of solutions is then traced using numerical methods, such as predictor-corrector algorithms. The challenge lies in carefully formulating the homotopy function and ensuring the path remains well-behaved, avoiding singularities or bifurcations that could make tracing difficult.

Conclusion

Infinitely split Nash equilibrium problems in repeated games represent a significant frontier in game theory, offering a rich yet complex domain for analysis. While traditional methods often fall short in providing constructive solutions or handling the multiplicity of equilibria, the homotopy method presents itself as a powerful and versatile tool. Its ability to continuously deform a simple problem into a complex one, constructively trace solution paths, and navigate non-linear landscapes makes it ideally suited for uncovering the intricate equilibrium structures of these dynamic games. As research in repeated games continues to evolve, the application and further development of homotopy methods will undoubtedly play a crucial role in deepening our understanding of strategic interactions over infinite horizons.

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