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Application of Fixed Point Theorems in Fuzzy Metric Space

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Abstract

In this paper we use rational inequality to prove fixed points theorems in fuzzy metric space. Our results extend the result of many other authors existing in the literature. Main aim of this paper is to show applications of fixed point theorem in fuzzy metric space.

Key-words: Fuzzy metric space, rational expression, fixed point

1. Introduction

The foundation of fuzzy mathematics is laid by Lofti A. Zadeh [3] with the introduction of fuzzy sets in 1965. This foundation represents vagueness in everyday life. Subsequently several authors have applied various forms of general topology of fuzzy sets and developed the concept of fuzzy space.In 1975, Kramosil and Michalek [5] introduced concept of fuzzy metric space. In 1988, Mariusz Grabiec [4] extended fixed point theorem of banach and eldestien to fuzzy metric spaces in the sense of Kramosil and Michalek [5]. In 1994, George et. al. [1] modified the notion of fuzzy metric spaces with the help of continuous t-norms. A number of fixed point theorem have been obtained by many authors by using the concept of compatible map, implicit relation, weakly compatible map, R-weakly compatible map [7 -14]. Also R. K. Saini and Vishal Gupta [9-10] proved some fixed point theorems in fuzzy metric space. The present paper extends the result of Mariusz Grabeic [4] and also many other authors existing in the literature.

2. Preliminaries

In this section, we define some definitions and results which are used in sequel.

Definition 2.1 [3] Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1]

Definition 2.2 [2] A binary operation $*$: [0, 1] x [0,1] → [0,1] is a continuous t-norms if ([0, 1], $*$) is an abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c, d ∈ [0, 1]

Definition 2.3 [5] A triplet (X, M, $*$) is a fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm and M is a fuzzy set on X^2 x (0, ∞) satisfying the following conditions, for all x, y, z ∈ X, such that t, s ∈ (0, ∞).

F1. M(x, y, z) > 0

F2. M(x, y, t) = 1 iff x = y

F3. M(x, y, t) = M(y, x, t)

F4. M(x, y, t) $*$ M(y, z, s) ≤ M(x, z, t + s)

F5. M(x, y, .) : [0, ∞) → [0,1] is continuous.

Then M is called a fuzzy metric on X and M(x, y, t) denotes the degree of nearness between x and y with respect to t.

Definition 2.4 [4] Let (X, M, $*$) is a fuzzy metric space then a sequence $\{x_n\}\in X$ is said to be convergent to a point x if $\lim_{n\to\infty} M(x_n, x, t) = 1$, ∀ t > 0

Definition 2.5 [4] Let (X, M, $*$) is a fuzzy metric space then a sequence $\{x_n\}\in X$ is called a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, ∀ t > 0 and p > 0

Definition 2.6 [4] Let (X, M, $*$) is a fuzzy metric space. If every Cauchy sequence is convergent in it, then it is called complete fuzzy metric space. If every sequence contains a convergent subsequence, then it is called compact

Lemma 2.7 [4] For all x, y ∈ X, M(x, y, .) is non-decreasing.

Lemma 2.8 [11] If there exist k ∈ (0, 1) such that M(x, y, kt) ≥ M(x, y, t), ∀ x, y ∈ X and t ∈ (0, ∞), then x = y

3. Main Result

We prove the following theorems.

Theorem 1 - Let (X, M,$*$) be a complete fuzzy metric space. and f : X → X be a mapping satisfying

$$M(x, y, t) = 1 \qquad (3.1)$$

And
$$M(fx, fy, kt) \geq \lambda(x, y, t) \qquad (3.2)$$

Where $\lambda(x,y,t)$=
$$min\left\{\frac{M(y,\ fy,\ t)\,[1+M(x,\ fx,\ t)]}{1+M(x,\ y,\ t)}, \frac{M(x,\ fx,\ t)\,[1+M(y,\ fy,\ t)]}{1+M(fx,\ f\,y,\ t)}, M(fx,fy,t), M(x,y,t)\right\}$$

(3.3)

for all x, y ∈ X and k ∈ (0, 1). Then f has a unique fixed point.

Proof- Let x ∈ X be any arbitrary point in X. Let us consider a sequence $\{x_n\}$ in X such that $fx_n = fx_{n+1}$, ∀ n ∈ N

First our aim is to show $\{x_n\}$is a Cauchy sequence.

Let $x = x_{n-1}$ and $y = x_n$ put in (3.2), we get

$M(fx_{n-1}, fx_n, kt) \geq \lambda(x_{n-1}, x_n, t)$

$\Rightarrow M(x_n, x_{n+1}, kt) \geq \lambda(x_{n-1}, x_n, t)$

Then $M(x_n, x_{n+1}, kt) = M(fx_{n-1}, fx_n, kt) \geq \lambda(x_{n-1}, x_n, t)$ (3.4)

Now $$\lambda(x_{n-1}, x_n, t) = min\left\{\begin{matrix}\frac{M(x_n, fx_n, t)\,[1+M(x_{n-1}, fx_{n-1}, t)]}{1+M(x_{n-1},\ x_n,\ t)}, & \frac{M(x_{n-1}, fx_{n-1}, t)\,[1+M(x_n, fx_n, t)]}{1+M(fx_{n-1},\ fx_n,\ t)}, \\ M(fx_{n-1},\ fx_n,\ t), & M(x_{n-1},\ x_n,\ t)\end{matrix}\right\}$$

$$= min\left\{\begin{matrix}\frac{M(x_n, x_{n+1}, t)\,[1+M(x_{n-1}, x_n, t)]}{1+M(x_{n-1},\ x_n,\ t)}, & \frac{M(x_{n-1}, x_n, t)\,[1+M(x_n, x_{n+1}, t)]}{1+M(x_n,\ x_{n+1},\ t)}, \\ M(x_n,\ x_{n+1},\ t), & M(x_{n-1},\ x_n,\ t)\end{matrix}\right\}$$

$$= \min\{M(x_n,\ x_{n+1},\ t), M(x_{n-1},\ x_n,\ t)\}$$

If $M(x_n,\ x_{n+1},\ t) \leq M(x_{n-1},\ x_n,\ t)$

$M(x_n, x_{n+1}, kt) \geq M(x_n,\ x_{n+1},\ t)$, by (3.4)

$\Rightarrow x_n = x_{n+1}$, by Lemma (2.8)

=> Sequence $\{x_n\}$ is a Cauchy sequence.

If $M(x_n,\ x_{n+1},\ t) \geq M(x_{n-1},\ x_n,\ t)$

Now by simple induction, for all n and t > 0

$M(x_n, x_{n+1}, kt) \geq M\left(x, x_1, \frac{t}{k^{n-1}}\right)$ (3.5)

Now for any positive integer 's' we have

$$M(x_n,\ x_{n+s},\ t) \geq M\left(x_n, x_{n+1}, \frac{t}{s}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{s}\right) * \ldots\ldots * M\left(x_{n+p-1}, x_{n+p}, \frac{t}{s}\right)$$

$$M(x_n,\ x_{n+s},\ t) \geq M\left(x, x_1, \frac{t}{sk^n}\right) * M\left(x, x_1, \frac{t}{sk^n}\right) * \ldots\ldots * M\left(x, x_1, \frac{t}{sk^n}\right)$$

By(3.5)

as n $\rightarrow \infty$, and using (3.1)

$\lim_{n\to\infty} M(x_n, x_{n+s}, t) = 1$, by (3.1)

=>Sequence $\{x_n\}$ is a Cauchy sequence.

Since (X, M, $*$) is a complete fuzzy metric space.

Then sequence $\{x_n\}$ is convergent in it.

Let $\{x_n\}$ converges to u ∈ (X, M, $*$) (3.6)

Now our aim is to show u is a fixed point of f.

Let us consider

$M(u, x_{n+1}, t) * M(x_{n+1}, fu, t) \leq M(u, fu, t)$, by(F5)

$M(u, fu, t) \geq M(u, x_{n+1}, t) * M(fx_n, fu, t)$

$M(u, fu, t) \geq M(u, x_{n+1}, t) * \lambda(x_n, u, \frac{t}{2k})$, by (3.4)

$= \lambda(x_n, u, \frac{t}{2k}) * M(u, x_{n+1}, t)$ (3.7)

Now $\lambda(x_n, u, \frac{t}{2k})$ =

$$min\left\{\begin{matrix} \frac{M\left(u, fu, \frac{t}{2k}\right)\left[1 + M\left(x_n, fx_n, \frac{t}{2k}\right)\right]}{1 + M\left(x_n, u, \frac{t}{2k}\right)}, & \frac{M\left(x_n, fx_n, \frac{t}{2k}\right)\left[1 + M\left(u, fu, \frac{t}{2k}\right)\right]}{1 + M\left(fx_n, fu, \frac{t}{2k}\right)}, \\ M\left(fx_n, fu, \frac{t}{2k}\right), & M\left(x_n, u, \frac{t}{2k}\right) \end{matrix}\right\}$$

as n $\rightarrow \infty$, by (3.1) and (3.7), we get

$\lambda(u, u, \frac{t}{2k}) = \min\left\{M\left(u, fu, \frac{t}{2k}\right), 1\right\}$ (3.8)

If $M\left(u, fu, \frac{t}{2k}\right) \geq 1$

=> $\lambda(u, u, \frac{t}{2k}) = 1$

Then by (3.7) and (2.3)

u is a fixed point of f.

If $M\left(u, fu, \frac{t}{2k}\right) \leq 1$

Then $\lambda(u, u, \frac{t}{2k}) = M\left(u, fu, \frac{t}{2k}\right)$, by (3.8)

Hence from (3.7)

$$M(u, fu, t) \geq M\left(u, fu, \frac{t}{2k}\right) * M(x_{n+1}, u, t) \qquad (3.9)$$

as $n \to \infty$ in (3.9) and using (3.1) and Lemma (2.8)

We get fu = u

Uniqueness

Now we show that u is an unique fixed point of f.

Let u and $v \in X$are two fixed points of f .

Then fu = u and fv = v

Consider

$$1 \geq M(v, u, t) = M(fu, fv, t) \geq \lambda(v, u, \frac{t}{k}) \qquad (3.10)$$

where

$\lambda(v, u, \frac{t}{k}) =$

$$min\left\{\frac{M\left(u,\ fu,\ \frac{t}{k}\right)[1+M\left(v,\ fv,\ \frac{t}{k}\right)]}{1+M\left(v,\ u,\ \frac{t}{k}\right)}, \frac{M\left(v,\ fv,\ \frac{t}{k}\right)[1+M\left(u,\ fu,\ \frac{t}{k}\right)]}{1+M\left(fv, fu,\ \frac{t}{k}\right)}, M\left(fv, fu, \frac{t}{k}\right), M(v, u, \frac{t}{k})\right\}$$

$$=min\left\{\frac{M\left(u,\ u,\ \frac{t}{k}\right)\left[1+M\left(v,\ v,\ \frac{t}{k}\right)\right]}{1+M\left(v,\ u,\ \frac{t}{k}\right)}, \frac{M\left(v,\ v,\ \frac{t}{k}\right)\left[1+M\left(u,\ u,\ \frac{t}{k}\right)\right]}{1+M\left(v,\ u,\ \frac{t}{k}\right)}, M\left(v, u, \frac{t}{k}\right), M\left(v, u, \frac{t}{k}\right)\right\}$$

$$= min\left\{\frac{1\,[1+1]}{1+M\left(v,\ u,\ \frac{t}{k}\right)}, \frac{1\,[1+1]}{1+M\left(v,\ u,\ \frac{t}{k}\right)}, M\left(v, u, \frac{t}{k}\right), M(v, u, \frac{t}{k})\right\}, \qquad \text{by (F2)}$$

$$= min\left\{\frac{1\,[1+1]}{1+1}, \frac{1\,[1+1]}{1+1}, 1, 1\right\}, \qquad \text{by (3.1)}$$

$= 1$

Then use in (3.10)

$1 \geq M(v, u, t) \geq 1$

$\Rightarrow M(v, u, t) = 1$

=> v = u

There exists unique fixed point of f.

Theorem 2 - Let (X, M,$*$) be a complete fuzzy metric space. and f : X → X be a mapping satisfying

$$M(x, y, t) = 1$$

and $$M(fx, fy, kt) \geq (\varphi_1\varphi_2 \ldots\ldots \varphi_n)[\lambda(x, y, t)] \quad , \quad (3.11)$$

Where $\lambda(x, y, t) =$
$$min\left\{\frac{M(y,\ fy,\ t)\,[1+M(x,\ fx,\ t)]}{1+M(x,\ y,\ t)}, \frac{M(x,\ fx,\ t)\,[1+M(y,\ fy,\ t)]}{1+M(fx,\ fy,\ t)}, M(fx, fy, t), M(x, y, t)\right\}$$

for all x, y ∈ X and k ∈ (0, 1), $\varphi_i \in \Psi$, for i = 1, 2, …, n

where Ψ is defined as Ψ = { φ, where φ : [0,1] → [0,1] } is continuous function such that $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi_i(a) \geq a$, for such 0 < a < 1 and i = 1, 2, …, n
(3.12)

Then f has a unique fixed point.

Proof – Since φ ∈ Ψ => $\varphi_i(a) \geq a$, for each 0 < a < 1 and i = 1,2,…,n, Then by (3.11)

$$M(fx, fy, kt) \geq (\varphi_1\varphi_2 \ldots\ldots \varphi_n)[\lambda(x, y, t)]$$

$$= (\varphi_1\varphi_2 \ldots\ldots \varphi_{n-1})\varphi_n\,[\lambda(x, y, t)]$$

$$\geq (\varphi_1\varphi_2 \ldots\ldots \varphi_{n-1})\lambda(x, y, t) \text{ , by (3.12)}$$

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$$\geq \lambda(x, y, t)\text{, by (3.12)}$$

=>$M(fx, fy, kt) \geq \lambda(x, y, t)$

Now applying Theorem-1, we get required result.

i.e. f has a unique fixed point.

Applications

In this section, we give some applications related to our results.

Let us define $\boldsymbol{\Phi}$: $[0, \infty) \to [0, \infty)$ and $\boldsymbol{\Phi}(t) = \int_0^t \phi(t)dt$, $\forall$ t > 0 be a non decreasing and continuous function. For each $\epsilon > 0$, $\phi(\epsilon) > 0$ Also implies that $\phi(t) = 0$ iff t = 0

Theorem 3 - Let (X, M,$*$) be a complete fuzzy metric space, and f : X → X be a mapping satisfying

$$M(x, y, t) = 1$$

$$\text{and } \int_0^{M(fx,fy,kt)} \phi(t)dt \geq \int_0^{\lambda(x,y,t)]} \phi(t)dt$$

Where $\lambda(x, y, t) =$
$$min\left\{\frac{M(y,\ fy,\ t)\,[1+M(x,\ fx,\ t)]}{1+M(x,\ y,\ t)}, \frac{M(x,\ fx,\ t)\,[1+M(y,\ fy,\ t)]}{1+M(fx,\ fy,\ t)}, M(fx, fy, t), M(x, y, t)\right\}$$

for all x, y ∈ X and k ∈ (0, 1), $\phi \in \boldsymbol{\Phi}$

Then f has a unique fixed point.

Proof – By taking $\phi(t) = 1$ and applying Theorem-1, we get the result.

Theorem 4 - Let (X, M,$*$) be a complete fuzzy metric space and f : X → X be a mapping satisfying

$$M(x, y, t) = 1$$

$$\text{and } \int_0^{M(fx,fy,kt)} \phi(t)dt \geq (\varphi_1\varphi_2 \ldots\ldots \varphi_n) \int_0^{\lambda(x,y,t)]} \phi(t)dt$$

Where $\lambda(x, y, t) =$
$$min\left\{\frac{M(y,\ fy,\ t)\,[1+M(x,\ fx,\ t)]}{1+M(x,\ y,\ t)}, \frac{M(x,\ fx,\ t)\,[1+M(y,\ fy,\ t)]}{1+M(fx,\ fy,\ t)}, M(fx, fy, t), M(x, y, t)\right\}$$

for all x, y ∈ X and k ∈ (0, 1), $\phi \in \boldsymbol{\Phi}$ and $\varphi_i \in \Psi$,for i = 1, 2, …., n

Then f has a unique fixed point.

Proof – Since φ_i (a) ≥ a , for any $\varphi_i \in \Psi$,for i = 1,2,….,n

Therefore $\varphi_1(a) \geq a$, $\varphi_2(a) \geq a$,…. , $\varphi_n(a) \geq a$ for each 0 < a < 1

$$\because \int_0^{M(fx,fy,kt)} \phi(t)dt \geq (\varphi_1\,\varphi_2\,\varphi_3 \ldots\ldots \varphi_n)\{\int_0^{\lambda(x,y,t)]} \phi(t)dt\}$$
$$= (\varphi_1\,\varphi_2\,\varphi_3 \ldots\ldots \varphi_{n-1})\varphi_n\,\{\int_0^{\lambda(x,y,t)]} \phi(t)dt\}$$
$$\geq (\varphi_1\,\varphi_2\,\varphi_3 \ldots\ldots \varphi_{n-1})\{\int_0^{\lambda(x,y,t)]} \phi(t)dt\}$$
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$$\geq \int_0^{\lambda(x,y,t)]} \phi(t)dt$$

Then by theorem-3, f has a unique fixed point

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