Primitive central idempotents of certain finite semisimple group algebras

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Abstract

The objective of this paper is to give a complete algebraic structure of semisimple group algebras of some finite indecomposable groups, whose central quotient is the Klein's four group, over a finite field.

Keywords: semisimple group algebra, metabelian groups , indecomposable groups, primitive central idempotents, Wedderburn decomposition.

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1. Introduction

Let $\mathbb{F}q$ be a finite field with q elements and G be a finite group of order coprime to q, so that the group algebra $\mathbb{F}_q[G]$ is semisimple. The most important problem in the area of group algebras is to find a complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$. The knowledge of primitive central idempotents is useful in finding Wederburn decomposition, unit group of integral group ring, various parameters in error correcting codes **[1,2,4,5,10,11,13,14,15,16,17,18]**. In **[3]**, Bakshi et.al. obtained a complete algebraic structure of $\mathbb{F}_q[G]$, G metabelian, using Strong Shoda pairs. They further illustrated their result by providing a complete set of primitive central idempotents and the Wedderburn decomposition of certain finite group algebras of indecomposable groups whose central quotient is Klein's four group. Further, Neha et. al. **[12]** obtained a complete Wedderburn decomposition of group algebras of all such indecomposable groups using the method developed by Ferraz in **[6]**. In this paper, we give a complete algebraic structure of $\mathbb{F}_q[G]$ for some indecomposable groups G, as classified by Milies **[7]**, using the method developed in **[3]**.

2. Metabelian groups

We recall the structure of metabelian group algebras over finite field as given in **[3]**.

Let $H \trianglelefteq K \trianglelefteq G$ with *K/H* cyclic of order *n*. Let $Irr(K/H)$ be the set of irreducible characters of *K/H* over the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $\mathcal{C}$ (*K/H*) be the set of *q*-cyclotomic cosets of $Irr(K/H)$ containing the generators of $Irr(K/H)$, i.e., if χ is a generator of $Irr(K/H)$, then an element *C* of $\mathcal{C}$(*K/H*) containing χ is the set $\{\chi, \chi^q, \ldots, \chi^{q^{o-1}}\}$, where $o = o_n(q)$, the order of *q* modulo *n*. Consider the action of $N_G(H)$, the normalizer of *H* in *G*, on $\mathcal{C}$(*K/H*) by conjugation. Let E_G(*K/H*) denote the stabilizer of $C \in \mathcal{C}(K/H)$ and $\mathcal{R}(K/H)$ denote the set of distinct orbits of $\mathcal{C}$(*K/H*) under this action. Set

$$\varepsilon_C(K,H) = |K|^{-1} \sum_{g \in K} tr_{\mathbb{F}_q(\zeta)/\mathbb{F}_q}(\chi(\bar{g}))\, g^{-1},$$

where χ is a representative of the *q*-cyclotomic coset *C* and ζ is a primitive *n*th root of unity in $\overline{\mathbb{F}}_q$, $C \in \mathcal{C}(K/H)$. Let $e_C(\mathrm{G,K,H})$ be the sum of distinct *G* -conjugates of $e_C(K,H)$. For a ring R, let $R^{(n)}$ denote the n-copies of R.

For a normal subgroup *N* of *G*, let A_N/N be a maximal abelian subgroup of *G/N* containing its derived subgroup $(G/N)'$. Let T be the set of all subgroups *D/N* of *G/N* with $D/N \leq A_N/N$ and A_N/D cyclic. Consider $T_{G/N}$ to be a set of representatives of the distinct equivalence classes of T under the equivalence relation of conjugacy in *G/N*. Define

$$S_{G/N} \coloneqq \{(D/N, A_N/N) \mid D/N \in T_{G/N},\ D/N \text{ core free in } G/N\} \text{ and}$$

$$S \coloneqq \{(N, D/N, A_N/N) \mid N \trianglelefteq \mathrm{G},\ S_{G/N} \neq \emptyset, (D/N, A_N/N) \in S_{G/N}\}$$

Theorem 1 [3] Let $\mathbb{F}_\mathrm{q}$ be a finite field with q elements and G a finite metabelian group. Suppose that $\gcd(q, |G|) = 1$. Then a complete set of primitive central idempotents of $\mathbb{F}_\mathrm{q}[\mathrm{G}]$ is given by the set

$$\{e_C(G, A_N, D) \mid (N, D/N, A_N/\mathrm{N}) \in S,\ C \in \mathcal{R}(A_N/\mathrm{D})\}.$$

Moreover, the corresponding simple component $F_q[G]e_c(G, A_N, D)$ is isomorphic to $M_{[G:A_N]}\left(\mathbb{F}_{q^{o(A_{N,D})}}\right)$, the algebra of $[G:A_N] \times [G:A_N]$ matrices over the field $F_{q^{o(A_{N,D})}}$, where $o(A_N, D) = \frac{o_{[A_N:D]}(q)}{[E_G(A_N/D):A_N]}$.

The groups *G* of the type $G/\mathbb{Z}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}(G)$ denotes the centre of the group *G*, are studied by Goodaire [**8,9**]. It has been proved in **[7]**, that the finite indecomposable groups with $G/\mathbb{Z}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ break into five classes as follows:

Group	Generators	Relations
D_1	x, y, t	$x^2, y^2, t^{2^m}, y^{-1}x^{-1}yxt^{2^{m-1}}, t$ central, $m \geq 1$
D_2	x, y, t	$x^2t^{-1}, y^2t^{-1}, t^{2^m}, y^{-1}x^{-1}yxt^{2^{m-1}}, t$ central, $m \geq 1$
D_3	a, b, x, y	$a^2, b^2y^{-1}, x^{2^{m_1}}, y^{2^{m_2}}, b^{-1}a^{-1}bax^{2^{m_1-1}}, x, y$ central, $m_1, m_2 \geq 1$
D_4	a, b, x, y	$a^2x^{-1}, b^2y^{-1}, x^{2^{m_1}}, y^{2^{m_2}}, b^{-1}a^{-1}bax^{2^{m_1-1}}, x, y$ central, $m_1, m_2 \geq 1$
D_5	a, b, x, y, z	$a^2y^{-1}, b^2z^{-1}, x^{2^{m_1}}, y^{2^{m_2}}, z^{2^{m_3}}, b^{-1}a^{-1}bax^{2^{m_1-1}}, x, y, z$ central, $m_1, m_2, m_3 \geq 1$

The complete algebraic structure of $\mathbb{F}_q[G]$, G of type D_1, D_2, is studied in [**3**]. In this paper, we will find the complete algebraic structure of $\mathbb{F}_q[G]$, G of type D_3.

3. Groups G of type D_3

$\mathrm{G} := \mathrm{D}_3 =< a, b, x, y \mid \mathrm{a}^2 = 1, \mathrm{b}^2 = 1, \mathrm{x}^{2^{m_1}} = \mathrm{y}^{2^{m_2}} = 1, \mathrm{a}^{-1}\mathrm{b}^{-1}\mathrm{ab} = \mathrm{x}^{2^{m_1-1}}, \mathrm{x}, \mathrm{y}$ central in $G >$

Theorem 2 For $m_1 = 1, m_2 \geq 1$ the complete algebraic structure of semisimple group algebra, $\mathbb{F}_q[G]$, G of type D_3, is given as follows:

Primitive Central Idempotents

$e_C(G, G, < x, a >),\ C \in \mathcal{R}(G/< x, a >);$

$e_C(G, G, < x, b >),\ C \in \mathcal{R}(G/< x, b >);$

$e_C\left(G, G, < x, a, b^{2^i} >\right),\ C \in \mathcal{R}(G/< x, a, b^{2^i} >);$

$e_C\left(G, G, < x, ab^{2^i} >\right),\ C \in \mathcal{R}(G/< x, ab^{2^i} >);$

$e_C\left(G, G, < x^2, x^ia, xb^{2^j} >\right),\ C \in \mathcal{R}(G/< x^2, x^ia, xb^{2^j} >),\ i, j = 0,1;$

$e_C\left(G, G, < x^{2^v}, x^ia, b >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^ia, b >);$

$e_C\left(G, G, < x^{2^v}, x^ia, x^jb >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^ia, x^jb >),\ 1 \leq v \leq m_1 - 1,\ i = 0, 2^{v-1},$
$1 \leq j \leq v - 1,\ \gcd(j, 2^v) \geq 2^{v-2};$

$e_C(G, < b, x >, < b >),\ C \in \mathcal{R}(< b, x >/< b >);$

$e_C\left(G, < a, x, y >, < a, x^{2^{m_1-1}}y >\right),\ C \in \mathcal{R}(< a, x, y >/< a, x^{2^{m_1-1}}y >).$

Wedderburn Decomposition

$$\mathbb{F}_q[G] \cong \mathbb{F}_q^{(8)} \oplus (\mathbb{F}_{q^{f_2}})^{(\frac{8}{f_2})} \oplus_{v=2}^{m_1-1} (\mathbb{F}_{q^{f_v}})^{(\frac{2^{v+2}}{f_v})} \oplus M_2(\mathbb{F}_{q^{f_{m_1}}})^{(\frac{2^{m_1}}{f_{m_1}})}$$

where $f_i = o_{2^i}(q)$, the order of q modulo 2^i, $i \geq 1$.

In order to find a complete set of primitive central idempotents and Wedderburn decomposition of $\mathbb{F}_q[G]$, G of type D_3, we need to find all the normal subgroups N of G and the corresponding $S_{G/N}$.

Lemma 1 All the distinct normal subgroups of G are as follows:

$\{e\}, < y >, < x^{2^{m_1-1}} y >$;

$< x >, < x, a >, < x, b^{2^i} >, < x, ab^{2^i} >, < x, a, b^{2^i} >, \quad i = 0,1$;

$< x^{2^v} >, < x^{2^v}, x^j a >, < x^{2^v}, x^j b^{2^i} >, < x^{2^v}, x^j ab^{2^i} >, < x^{2^v}, x^j a, b^{2^i} >$,

$< x^{2^v}, x^j a, x^{2^{v-1}} b^{2^i} >, \quad 1 \leq v \leq m_1 - 1,\ j = 0, 2^{v-1}, i = 0, 1$;

$< x^{2^v}, x^j b >, < x^{2^v}, x^j ab >, < x^{2^v}, x^j a, x^i b >, \ 2 \leq v \leq m_1 - 1,\ j = 0, 2^{v-1}$,

$i = 2^{v-2}, 3.2^{v-2}$.

Proof: Let $N \trianglelefteq G$ be such that $N \cap < x >= \{e\}$, then $N = < y >$ or $< x^{2^{m_1-1}} y >$ or $\{e\}$. For $N \cap < x >=< x >$, it is easy to see that N is either $< x >$ or $< x, a >$ or $< x, b^{2^i} >$ or $< x, ab^{2^i} >$ or $< x, a, b^{2^i} >, i = 0, 1$.

Let us assume that $N \cap < x >=< x^{2^v} >$, $1 \leq v \leq m_1 - 1$. Now, $N/< x^{2^v} >$ is isomorphic to one of the following: $< x >, < a < x >>, < b^{2^i} < x >>, < ab^{2^i} < x >>$, or $< a < x >, b^{2^i} < x >>, 0 \leq i \leq 1$. Let $N/< x^{2^v} >$ be isomorphic to $< x >$, then $N = < x^{2^v} >$, further if $N/< x^{2^v} >$ is isomorphic to $< a < x >>$ then N is either $< x^{2^v}, a >$ or $< x^{2^v}, x^j a >,\ 1 \leq j \leq 2^{v-1}$. But if $\gcd(j, 2^v) = 2^\alpha$, then $(x^{2^\alpha} a)^2 = x^{2^{\alpha+1}}$ which will lie in $< x^{2^v}, x^j a >$ if, and only if, $\alpha \geq v - 1$. Thus $j = 2^{v-1}$, hence in this case , $N =< x^{2^v}, a >$ or $< x^{2^v}, x^{2^{v-1}} a >$.

Now, if $N/< x^{2^v} > \cong< b < x >>$, then for $v = 1, N/< x^2 > \cong< b < x >>$, and $N =< x^2, b >$ or $< x^2, xb >$. Similarly, for $2 \leq v \leq m_1 - 1$, we have N is either $< x^{2^v}, b >$ or $< x^{2^v}, x^j b >,\ 1 \leq j \leq 2^{v-1}$. Let $\gcd(j, 2^v) = 2^\alpha$, then $(x^{2^\alpha} b)^4 = x^{2^{\alpha+2}}$, which will lie in $< x^{2^v}, x^j b >$ if, and only if, $\alpha \geq v - 2$. Thus, for $2 \leq v \leq m_1 - 1$, $N = < x^{2^v}, x^{2^{v-2}} b >$ or $< x^{2^v}, x^{3.2^{v-2}} b >$ or $< x^{2^v}, x^{2^{v-1}} b >$.

Similarly for $N/< x^{2^v} > \cong< b^2 < x >>$, either $N =< x^2, b^2 >$ or $< x^2, xb^2 >$ or $< x^{2^v}, b^2 >$ or $< x^{2^v}, x^{2^{v-1}} b^2 >$, $2 \leq v \leq m_1 - 1$. Next, for $N/< x^{2^v} > \cong< ab < x >>$, either $N =< x^2, ab >$ or $< x^2, xab >$ or $< x^{2^v}, ab >$ or $< x^{2^v}, x^{2^{v-1}} ab >$ or $< x^{2^v}, x^{2^{v-2}} ab >$ or $< x^{2^v}, x^{3.2^{v-2}} ab >$, $2 \leq v \leq m_1 - 1$.

Further for $N/<x^{2^v}> \cong <ab^2<x>>$, either $N=<x^{2^v}, ab^2>$ or $<x^{2^v}, x^{2^{v-1}}ab^2>$, $1 \le v \le m_1-1$ Next, for $N/<x^{2^v}> \cong <a<x>, b<x>>$, $N=<x^2, a, xb>$, $<x^2, xa, xb>$, $<x^{2^v}, a, b>$, $<x^{2^v}, x^{2^{v-1}}a, b>$, $1 \le v \le m_1-1$, $<x^{2^v}, x^i a, x^{2^{v-2}}b>$, $<x^{2^v}, x^i a, x^{3.2^{v-2}}b>$, $<x^{2^v}, x^i a, x^{2^{v-1}}b>$, $2 \le v \le m_1-1$, $i=0, 2^{v-1}$. Finally, if $N/<x^{2^v}> \cong <a<x>, b^2<x>>$, then $N=<x^{2^v}, a, b^2>$, $<x^{2^v}, x^{2^{v-1}}a, b^2>$, $<x^{2^v}, x^i a, x^{2^{v-1}}b^2>$, $1 \le v \le m_1-1$, $i=0, 2^{v-1}$.

Observe that if $N \cap <x> = <x^{2^v}>$, $0 \le v \le m_1-1$ then $G' = <x^{2^{m_1-1}}> \subseteq N$ and hence G/N is abelian. Thus,

$$S_{G/N} = \begin{cases} (<1>, G/N), & if\ G/N\ is\ cyclic, \\ \emptyset, & otherwise. \end{cases}$$

Out of the normal subgroups N, listed above, the following have cyclic quotient with G:
$<x, a>, <x, b>, <x, a, b^{2^i}>, <x, ab^{2^i}>$, $i=0,1$;
$<x^{2^v}, x^j a, x^{2^{v-2}}b>, <x^{2^v}, x^j a, x^{3.2^{v-2}}b^2>$, $2 \le v \le m_1-1$, $j=0, 2^{v-1}$;
$<x^{2^v}, x^j a, x^{2^{v-1}}b>, <x^{2^v}, x^j a, b>$, $1 \le v \le m_1-1$, $j=0, 2^{v-1}$;
$<x^2, a, xb^2>, <x^2, xa, xb^2>$.
Further, if $N=\{e\}$, then $S_{G/N}=\emptyset$, whereas for $N=<y>$,
$S_{G/N} = \{(<b>/N, <b, x>/N)\}$, and for $N=<x^{2^{m_1-1}}y>$,
$S_{G/N} = \{(<a, x^{2^{m_1-1}}y>/N, <a, x, y>/N)\}$.
Hence, the primitive central idempotents of $\mathbb{F}_q[G]$, as stated in Theorem 2, are obtained with the help of Theorem 1.
The Wedderburn decomposition of $\mathbb{F}_q[G]$ can now be easily obtained with the help of following table and Theorem 1.

N	(D, A_N)	$o(A_N, D)$	$\lvert\mathcal{R}(A_N, D)\rvert$
$<x, a>$	(N, G)	f_2	$\frac{2}{f_2}$
$<x, b>$	(N, G)	1	1
$<x, a, b^{2^i}>$, $0 \le i \le 1$	(N, G)	1	1
$<x, ab>$	(N, G)	1	1
$<x, ab^2>$	(N, G)	f_2	$\frac{2}{f_2}$
$<x^{2^v}, x^j a, x^{2^{v-1}}b>$, $j=0, 2^{v-1}$,	(N, G)	f_v	$\frac{2^{v-1}}{f_v}$

$1 \le v \le m_1 - 1$			
$< x^{2^v}, x^j a, b >$, $j = 0, 2^{v-1}$, $1 \le v \le m_1 - 1$	(N, G)	f_v	$\frac{2^{v-1}}{f_v}$
$< x^2, x^i a, xb^2 >$, $i = 0, 1$	(N, G)	f_2	$\frac{2}{f_2}$
$< x^{2^v}, x^j a, x^{2^{v-2}} b >$, $j = 0, 2^{v-1}$, $2 \le v \le m_1 - 1$	(N, G)	f_v	$\frac{2^{v-1}}{f_v}$
$< x^{2^v}, x^j a, x^{3.2^{v-2}} b >$, $j = 0, 2^{v-1}$, $2 \le v \le m_1 - 1$	(N, G)	f_v	$\frac{2^{v-1}}{f_v}$
$< y >$	$(< b >, < b, x >)$	f_{m_1}	$\frac{2^{m_1-1}}{f_{m_1}}$
$< x^{2^{m_1-1}} y >$	$(< a, x^{2^{m_1-1}} y >, < a, x, y >)$	f_{m_1}	$\frac{2^{m_1-1}}{f_{m_1}}$

Theorem 3 For $m_1 = 1, m_2 \ge 1$ the complete algebraic structure of semisimple group algebra, $\mathbb{F}_q[G]$, G of type D_3, is given as follows:

Primitive Central Idempotents

$e_C(G, G, < x, b >),\ C \in \mathcal{R}(G/< x, b >);$

$e_C\left(G, G, < x, ab^{2^i} >\right),\ C \in \mathcal{R}(G/< x, ab^{2^i} >),\ 0 \le i \le m_2;$

$e_C\left(G, G, < x, a, b^{2^i} >\right),\ C \in \mathcal{R}(G/< x, a, b^{2^i} >),\ 0 \le i \le m_2 + 1;$

$e_C\left(G, < a, x, y >, < a, xy^{2^i} >\right),\ C \in \mathcal{R}(< a, x, y >/< a, xy^{2^i} >),\ 0 \le i \le m_2 - 1;$

$e_C(G, < b, x >, < b >),\ C \in \mathcal{R}(< b, x >/< b >).$

Wedderburn Decomposition

$$\mathbb{F}_q[G] \cong \mathbb{F}_q{}^{(4)} \oplus_{i=1}^{m_2} \left(\mathbb{F}_{q^{f_{i+1}}}\right)^{\left(\frac{2^i}{f_{i+1}}\right)} \oplus_{i=2}^{m_2+1} \left(\mathbb{F}_{q^{f_i}}\right)^{\left(\frac{2^{i-1}}{f_i}\right)} \oplus M_2(\mathbb{F}_q) \oplus_{i=0}^{m_2-1} M_2(\mathbb{F}_{q^{f_{i+1}}})^{\left(\frac{2^i}{f_{i+1}}\right)}$$

where $f_i = o_{2^i}(q)$.

Proof: In order to prove this, we need to find all the distinct normal subgroups of G. Let N be a normal subgroup of G such that $N \cap < x >= \{e\}$ then clearly N is $\{e\}$ or $< y^{2^i} >$ or $< xy^{2^i} >$

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$, 0 \leq i \leq m_2 - 1$. Now let us assume that $N \cap < x > = < x >$, then $N / N \cap < x >$ is isomorphic to $< x >$ or $< a < x >>$ or $< b^{2^i} < x >>$ or $< ab^{2^i} < x >>$ or $< a < x >, b^{2^i} < x >>$, $0 \leq i \leq m_2$. Let $N / < x > \cong < x >$, thus $N \cong < x >$. If $N / < x > \cong < a < x >>$, then $N \cong < x, a >$. Similarly in other cases N will be one of the following $< x, ab^{2^i} >, < x, b^{2^i} >, < x, a, b^{2^i} >, 0 \leq i \leq m_2$.

Observe that if $N \cap < x > = < x >$, then G/N is abelian and hence

$$S_{G/N} = \begin{cases} (< 1 >, G/N), & \text{if } G/N \text{ is cyclic,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

It can again be seen easily that for $N \trianglelefteq G, N \cap < x > = < x >$, the following have cyclic quotients with G:

$< x, a >, < x, b >, < x, ab^{2^i} >, < x, a, b^{2^i} >$, $0 \leq i \leq m_2$.

Also observe that for $N = \{e\}, < y^{2^i} >, 1 \leq i \leq m_2 - 1$, $S_{G/N} = \emptyset$, whereas for $N = < y >$, $S_{G/N} = \{(< b > /N, < b, x > /N)\}$ and for $N = < xy^{2^i} >$, $0 \leq i \leq m_2 - 1$, $S_{G/N} = \{(< a, xy^{2^i} > /N, < a, x, y > /N)\}$.

The primitive central idempotents stated in Theorem 3 are thus obtained with the help of Theorem 1.

To find the Wedderburn decomposition of $\mathbb{F}_q[G]$, we compute the required parameters $o(A_N, D)$ and $|\mathcal{R}(A_N, D)|$ as follows:

N	$S_{G/N}$	$o(A_N, D)$	$\lvert\mathcal{R}(A_N, D)\rvert$
$< x, a, b >$	$\{(< 1 >, < 1 >)\}$	1	1
$< x, b >$	$\{(< 1 >, G/N)\}$	1	1
$< x, a, b^{2^i} >$, $1 \leq i \leq m_2 + 1$	$\{(< 1 >, G/N)\}$	f_i	$\frac{2^{i-1}}{f_i}$
$< x, ab^{2^i} >$, $1 \leq i \leq m_2$	$\{(< 1 >, G/N)\}$	f_{i+1}	$\frac{2^i}{f_{i+1}}$
$< y >$	$\{(< b > /N, < b, x > /N)\}$	1	1
$< xy^{2^i} >$, $1 \leq i \leq m_2 - 1$	$\{(< a, xy^{2^i} > /N, < a, x, y > /N)\}$	f_{i+1}	$\frac{2^i}{f_{i+1}}$

With the help of this table, the Wedderburn decomposition of $\mathbb{F}_q[G]$, as stated in Theorem 3, is obtained.

The proof of the following theorem is similar to the previous one, so we omit the details here.

Theorem 4 Let $m_1, m_2 > 1$. Then (i) For $m_1 = m_2$, the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ is given as:

Primitive Central Idempotents

$e_C(G, G, < x, a >),\ C \in \mathcal{R}(G/< x, a >);$

$e_C(G, G, < x, b >),\ C \in \mathcal{R}(G/< x, b >);$

$e_C\left(G, G, < x, a, b^{2^i} >\right),\ C \in \mathcal{R}\left(G/< x, a, b^{2^i} >\right),\ 1 \le i \le m_1;$

$e_C\left(G, G, < x, ab^{2^i} >\right),\ C \in \mathcal{R}\left(G/< x, ab^{2^i} >\right),\ 1 \le i \le m_1;$

$e_C(G, G, < x^{2^v}, x^j a, b >),\ C \in \mathcal{R}(G/< x^{2^v}, x^j a, b >),\ 1 \le v \le m_1 - 1,\ j = 0, 2^{v-1};$

$e_C(G, G, < x^{2^v}, x^k a, x^j b >),\ C \in \mathcal{R}(G/< x^{2^v}, x^k a, x^j b >),\ 1 \le v \le m_1 - 1,$
$\gcd(j, 2^v) \ge 1,\ k = 0, 2^{v-1};$

$e_C\left(G, G, < x^{2^v}, x^k a, x^j b^{2^\beta} >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^k a, x^j b^{2^\beta} >),\ 1 \le v \le m_1 - 1,$
$\gcd(j, 2^v) = 1,\ 1 \le \beta \le m_1 + 1 - v,\ \ k = 0, 2^{v-1};$

$e_C(G, < b, x >, < b >),\ C \in \mathcal{R}(< b, x >/< b >);$

$e_C(G, < a, x, y >, < a, x^j y >),\ C \in \mathcal{R}(< a, x, y >/< a, x^j y >),\ \ \gcd(j, 2^v) \ge 1.$

Wedderburn Decomposition

$$\mathbb{F}_q[G] \cong \mathbb{F}_q{}^{(6)} \oplus_{i=2}^{m_1+1} \left(\mathbb{F}_{q^{f_i}}\right)^{\left(\frac{2^i}{f_i}\right)} \oplus_{v=2}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\left(\frac{2^v}{f_v}\right)} \oplus_{v=1}^{m_1-1} \oplus_{\beta=0}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\left(\frac{2^{2v-\beta-1}}{f_v}\right)}$$
$$\oplus_{v=1}^{m_1-1} \oplus_{\beta=1}^{m_1+1-v} \left(\mathbb{F}_{q^{f_{\beta+v}}}\right)^{\left(\frac{2^{2v+\beta-1}}{f_{\beta+v}}\right)} \oplus M_2(\mathbb{F}_{q^{f_{m_1}}})^{\left(\frac{2^{m_1-1}}{f_{m_1}}\right)(2^{m_1})}.$$

(ii) For $m_1 > m_2$, the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ is given as:

Primitive Central Idempotents

$e_C(G, G, < x, a >),\ C \in \mathcal{R}(G/< x, a >);$

$e_C(G, G, < x, b >),\ C \in \mathcal{R}(G/< x, b >);$

$e_C\left(G, G, < x, a, b^{2^i} >\right),\ C \in \mathcal{R}\left(G/< x, a, b^{2^i} >\right),\ 1 \le i \le m_2;$

$e_C\left(G, G, < x, ab^{2^i} >\right),\ C \in \mathcal{R}\left(G/< x, ab^{2^i} >\right),\ 1 \le i \le m_2;$

$e_C\left(G, G, < x^{2^v}, x^j a, b >\right),\ C \in \mathcal{R}\left(G/< x^{2^v}, x^j a, b >\right),\ 1 \le v \le m_1 - 1,\ j = 0, 2^{v-1};$

$e_C\left(G, G, < x^{2^v}, x^k a, x^j b >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^k a, x^j b >),\ 1 \le v \le m_1 - 1,$
$\gcd(j, 2^v) \ge \max\{1, 2^{v-m_2-1}\},\ k = 0, 2^{v-1};$

$e_C\left(G, G, < x^{2^v}, x^k a, x^j b^{2^\beta} >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^k a, x^j b^{2^\beta} >),\ 1 \le v \le m_1 - 1,$
$\gcd(j, 2^v) = 1,\ 1 \le \beta \le m_2 + 1 - v,\ \ k = 0, 2^{v-1};$

$e_C(G, < b, x >, < b >),\ C \in \mathcal{R}(< b, x > / < b >);$

$e_C\left(G, < a, x, y >, < a, x^j y >\right),\ C \in \mathcal{R}\left(< a, x, y > / < a, x^j y >\right),\ \gcd(j, 2^v) \ge$
$\max\{1, 2^{m_1-m_2}\}.$

Wedderburn Decomposition

$$\mathbb{F}_q[G] \cong \mathbb{F}_q^{(6)} \oplus_{i=2}^{m_2+1} \left(\mathbb{F}_{q^{f_i}}\right)^{\left(\frac{2^i}{f_i}\right)} \oplus_{v=2}^{m_1-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\left(\frac{2^v}{f_v}\right)} \oplus_{v=1}^{m_2} \oplus_{\beta=0}^{v-1} \left(\mathbb{F}_{q^{f_v}}\right)^{\left(\frac{2^{2v-\beta-1}}{f_v}\right)}$$
$$\oplus_{v=m_2+1}^{m_1-1} \oplus_{\beta=v-m_2-1}^{v-1} \left(\mathbb{F}_{q^{f_2}}\right)^{\left(\frac{2^{2v-\beta-1}}{f_v}\right)} \oplus_{v=1}^{m_2} \oplus_{\beta=1}^{m_2+1-v} \left(\mathbb{F}_{q^{f_{\beta+v}}}\right)^{\left(\frac{2^{2v+\beta-1}}{f_{\beta+v}}\right)}$$
$$\oplus\ M_2(\mathbb{F}_{q^{f_{m_1}}})^{\left(\frac{2^{m_1-1}}{f_{m_1}}\right)(2^{m_2})}.$$

(ii) For $m_1 < m_2$, the complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ is given as:

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$e_C(G, G, < x, a >),\ C \in \mathcal{R}(G/< x, a >);$

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$e_C\left(G, G, < x, a, b^{2^i} >\right),\ C \in \mathcal{R}\left(G/< x, a, b^{2^i} >\right),\ 1 \le i \le m_2;$

$e_C\left(G, G, < x, ab^{2^i} >\right),\ C \in \mathcal{R}\left(G/< x, ab^{2^i} >\right),\ 1 \le i \le m_2;$

$e_C\left(G, G, < x^{2^v}, x^j a, b >\right),\ C \in \mathcal{R}\left(G/< x^{2^v}, x^j a, b >\right),\ \ j = 0, 2^{v-1};$

$e_C\left(G, G, < x^{2^v}, x^k a, x^j b >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^k a, x^j b >),\ 1 \le v \le m_1 - 1,$
$\gcd(j, 2^v) \ge \max\{1, 2^{v-m_2-1}\},\ k = 0, 2^{v-1};$

$e_C\left(G, G, < x^{2^v}, x^k a, x^j b^{2^\beta} >\right),\ C \in \mathcal{R}(G/< x^{2^v}, x^k a, x^j b^{2^\beta} >),\ 1 \le v \le m_1 - 1,$
$\gcd(j, 2^v) = 1,\ 1 \le \beta \le m_2 + 1 - v,\ \ k = 0, 2^{v-1};$

$e_C(G, < b, x >, < b >),\ C \in \mathcal{R}(< b, x >/< b >);$
$e_C\big(G, < a, x, y >, < a, x^j y >\big),\ C \in \mathcal{R}\big(< a, x, y >/< a, x^j y >\big),\ \ \gcd(j, 2^v) = 2^\beta,$
$0 \le \beta \le v - 1;$
$e_C\left(G, < a, x, y >, < a, x^j y^{2^\beta} >\right),\ C \in \mathcal{R}\left(< a, x, y >/< a, x^j y^{2^\beta} >\right),\ \ \gcd(j, 2^v) = 1,$
$1 \le \beta \le m_2 - m_1.$

Wedderburn Decomposition

$$\mathbb{F}_q[G] \cong \mathbb{F}_q{}^{(6)} \oplus_{\mathrm{i}=2}^{\mathrm{m}_2+1} \left(\mathbb{F}_{\mathrm{q}^{\mathrm{f_i}}}\right)^{\left(\frac{2^{\mathrm{i}}}{\mathrm{f_i}}\right)} \oplus_{\mathrm{v}=2}^{\mathrm{m}_1-1} \left(\mathbb{F}_{\mathrm{q}^{\mathrm{f_v}}}\right)^{\left(\frac{2^{\mathrm{v}}}{\mathrm{f_v}}\right)} \oplus_{\mathrm{v}=1}^{\mathrm{m}_1-1} \oplus_{\beta=0}^{\mathrm{v}-1} \left(\mathbb{F}_{\mathrm{q}^{\mathrm{f_v}}}\right)^{\left(\frac{2^{2\mathrm{v}-\beta-1}}{\mathrm{f_v}}\right)}$$
$$\oplus_{\mathrm{v}=1}^{\mathrm{m}_1-1} \oplus_{\beta=1}^{\mathrm{m}_2+1-\mathrm{v}} \left(\mathbb{F}_{\mathrm{q}^{\mathrm{f}_{\beta+\mathrm{v}}}}\right)^{\left(\frac{2^{2\mathrm{v}+\beta-1}}{\mathrm{f}_{\beta+\mathrm{v}}}\right)} \oplus \mathrm{M}_2(\mathbb{F}_{\mathrm{q}^{\mathrm{f}_{\mathrm{m}_1}}})^{\left(\frac{2^{\mathrm{m}_1-1}}{\mathrm{f}_{\mathrm{m}_1}}\right)(2^{\mathrm{m}_1})}$$
$$\oplus_{\beta=1}^{\mathrm{m}_2-\mathrm{m}_1} \mathrm{M}_2(\mathbb{F}_{\mathrm{q}^{\mathrm{f}_{\mathrm{m}_1+\beta}}})^{\left(\frac{2^{2\mathrm{m}_1+\beta-2}}{\mathrm{f}_{\mathrm{m}_1+\beta}}\right)}.$$

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