ON L^1-CONVERGENCE OF MODIFIED COSINE SUMS WITH PARTICULAR COEFFICIENTS

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Abstract. In this paper we introduced modified cosine sums and study its L^1-convergence and integrability under the class K of coefficients. It is shown that such type of modified cosine sums converge to the trigonometric series in L^1 – metric without any additional condition where as the classical trigonometric partial sums may require additional conditions for L^1 – convergence. Also a necessary and sufficient condition for the L^1 – convergence of cosine series has been deduced as a corollary under the said class.

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1 Introduction

If a trigonometric series converges in L^1 – norm to a function $f \in L^1$, then it is the Fourier series of the function f but the converse of this is not true. Riesz([1], Vol.II, Chap.VIII) gave a counter example that a Fourier series of a function f may not converge to f in L^1-norm. This motivated many authors to study L^1-convergence of trigonometric series by taking modified cosine and sine sums. It has been observed that these modified sum approximate their limits in the sense that they converge in L^1-norm where as classical sums may not or they may require some additional condition.

Many authors like Garrett and Stanojevič [6, 7], Kumari and Ram [13], Bor [2, 3], Chen [4], Kaur, Bhatia and Ram [9] and Krasniqui [11, 12] introduced modified trigonometric sums and studied their integrability and L^1-convergence under various classes.

Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k coskx \tag{1.1}$$

And

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k coskx \tag{1.2}$$

be the cosine series and its sum. Using modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j) coskx$$

ofGarett and Stanojević [5], Kaur and Bhatia [8] proved the following theorem under the class of generalized semi-convex coefficients:

TheoremA. If $\{a_n\}$ is a generalized semi-convex null sequence, then $g_n(x)$ converges to $f(x)$ in L^1-metrix if and only if $\Delta a_n \log n = o(1)$, as $n \to \infty$.

Kumari and Ram [13] also introduced modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\sin kx$$

and studied their L^1-convergence under the condition that the coefficient sequences $\{a_k\}$ belong to the classes S and R. They also deduced the results about L^1-convergence of cosine and sine series as corollaries.

Kaur [9] defined a new class K of coefficients in the following way:

Definition: Let k be a positive real number. If

$$a_k = o(1),\ k \to \infty \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} k|\Delta^2 a_{k-1} - \Delta^2 a_{k+1}| < \infty, \tag{1.4}$$

Any sequence satisfying (1.3) and (1.4) is said to belong to class **K**[9]. Also in [9] the class **K** is generalized in the following way:

Definition: Let α be a positive real number. If (1.3) holds and

$$\sum_{k=1}^{\infty} k^{\alpha}|\Delta^{\alpha+1} a_{k-1} - \Delta^{\alpha+1} a_{k+1}| < \infty, (a_0 = 0)$$

then we say that $\{a_n\}$ belongs to the class $\mathbf{K}^{\alpha}$

For α = 1, the class $\mathbf{K}^{\alpha}$ reduces to the class **K** and the following theorem is proved.

TheoremB. If $\{a_n\}$ belongs to the class **K**, then the necessary and sufficient conditions for L^1 –convergence of the cosine series (1.1) is $\lim_{n\to\infty} a_n \log n = 0$.

Again in Kaur [10], the following modified sums are introduced

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx$$

and proved the following result:

TheoremC. Let the sequence $\{a_k\}$ belongs to class $\mathbf{K}^{\alpha}$. Then $K_n(x)$ converges to $f(x)$ in L^1-norm.

In this paper we introduce the following modified cosine sums

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_{j-1} - a_{j+1}}{j}\right) k\cos kx$$

and study its integrability and L^1-convergence under the class **K** and deduce the necessary and sufficient condition. Our result can also be proved further under the generalized class K^{α}. In our main result, we use the concepts of **Dirichlet's Kernel and conjugate Dirichlet's Kernel [14]**. The nth partial sums of the series

$$\frac{1}{2}+\sum_{k=1}^{\infty} coskx \text{ and } \sum_{k=1}^{\infty} sinkx$$

Denoted by $D_n(x)$ and $\widetilde{D}_n(x)$ are called Dirichlet and conjugate Dirichlet Kernels respectively.

Thus

$$D_n(x)=\frac{1}{2}+cosx+cos2x+\cdots+cosnx=\frac{sin\left(n+\frac{1}{2}\right)x}{2sin\frac{x}{2}}$$

$$\widetilde{D}_n(x)=sinx+sin2x+\cdots+sinnx=\frac{cos\frac{x}{2}-cos\left(n+\frac{1}{2}\right)x}{2sin\frac{x}{2}}$$

If $x \neq 0 \pmod{2\pi}$, then

$$|D_n(x)\leq\frac{\pi}{2x}, for\ 0<|x|\leq\pi, and|\widetilde{D}_n(x)|\leq\frac{\pi}{x}, for\ 0<|x|\leq\pi$$

Also we use the uniform estimate

$$|D_n(x)|\leq n+\frac{1}{2}, \text{for any x, and the estimate}$$

$$L_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|D_n(x)|dx\approx\frac{4}{\pi^2}logn \text{ for Lebesgue constant}$$

We have similarly

$$\widetilde{L}_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|\widetilde{D}_n(x)|dx\approx logn$$

Fejér Kernel [14] The Fejér Kernel is defined as

$$K_n(x)=\frac{1}{n+1}\sum_{j=0}^{n}D_j(x),$$

It has the following properties;

(i) $K_n(x)\geq 0$, (ii) $\frac{1}{\pi}\int_{-\pi}^{\pi}|K_n(x)|=1$

The conjugate Fejér Kernel is defined as

$$\widetilde{K}_n(x)=\frac{1}{n+1}\sum_{j=0}^{n}\widetilde{D}_j(x)$$

Also we have $\widetilde{K}_n(x)>0$ for $0<x<\pi$ and $|\widetilde{K}_n(x)|<\frac{n}{2}$ for n = 1, 2, 3

Also we have the important result

$$\widetilde{D}'_n(x)=(n+1)D_n(x)-(n+1)K_n(x)$$

2 Main Result

Theorem. If $\{a_n\}$ belongs to the class **K**. Then $||f-g_n||=o(1)$, $n\rightarrow\infty$

Proof Consider the modified sums

$$g_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left(\frac{a_{j-1}-a_{j+1}}{j}\right)kcoskx$$

$$=\frac{a_0}{2}+\sum_{k=1}^{n}\left[\Delta\left(\frac{a_{k-1}-a_{k+1}}{k}\right)+\Delta\left(\frac{a_k-a_{k+2}}{k+1}\right)+\Delta\left(\frac{a_{k+1}-a_{k+3}}{k+2}\right)+\cdots+\Delta\left(\frac{a_{n-1}-a_{n+1}}{n}\right)\right]kcoskx$$

$= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\left(\frac{a_{k-1}-a_{k+1}}{k} - \frac{a_k - a_{k+2}}{k+1}\right) + \left(\frac{a_k - a_{k+2}}{k+1} - \frac{a_{k+1}-a_{k+3}}{k+2}\right) + \cdots + \left(\frac{a_{n-1}-a_{n+1}}{n} - \right.\right.$
an–an+2n+1kcoskx

$= \frac{a_0}{2} + \sum_{k=1}^{n}\left[\left(\frac{a_{k-1}-a_{k+1}}{k}\right) - \left(\frac{a_n - a_{n+2}}{n+1}\right)\right] kcoskx$

$= \frac{a_0}{2} + \sum_{k=1}^{n}(a_{k-1} - a_{k+1})coskx - \left(\frac{a_n - a_{n+2}}{n+1}\right)\sum_{k=1}^{n} kcoskx$

$= \frac{a_0}{2} + \sum_{k=1}^{n}(a_{k-1} - a_{k+1})coskx - \left(\frac{a_n - a_{n+2}}{n+1}\right)\tilde{D}'_n(x)$

$= S_n(x) - \left(\frac{a_n - a_{n+2}}{n+1}\right)\tilde{D}'_n(x)$ ......(2.1)

Applying Abel's Transformation, we get

$= \sum_{k=1}^{n-1}\Delta(a_{k-1} - a_{k+1})D_k(x) + (a_{n-1} - a_{n+1})D_n(x) - \left(\frac{a_n - a_{n+2}}{n+1}\right)\tilde{D}'_n(x)$

$= \sum_{k=1}^{n}\Delta(a_{k-1} - a_{k+1})D_k(x) + (a_n - a_{n+2})D_n(x) - \left(\frac{a_n - a_{n+2}}{n+1}\right)\tilde{D}'_n(x)$

Again applying Abel's transformation, we get

$= \sum_{k=1}^{n-1}\Delta^2(a_{k-1} - a_{k+1})(k+1)K_k(x) + (n+1)(\Delta a_{n-1} - \Delta a_{n+1})K_n(x) + (a_n -$
an+2Dnx–an–an+2n+1D′nx

$= \sum_{k=1}^{n}\Delta^2(a_{k-1} - a_{k+1})(k+1)K_k(x) + (n+1)(\Delta a_n - \Delta a_{n+2})K_n(x) + (a_n -$
an+2Dnx–an–an+2n+1D′nx

Using the result $\tilde{D}'_n(x)$ = (n+1)D_n(x) – (n+1)K_n(x), we get

$= \sum_{k=1}^{n}\Delta^2(a_{k-1} - a_{k+1})(k+1)K_k(x) + (n+1)(\Delta a_n - \Delta a_{n+2})K_n(x) + (a_n -$
an+2Dnx–an–an+2n+1[(n+1)*Dnx*– (n+1)*Knx]*

$= \sum_{k=1}^{n}\Delta^2(a_{k-1} - a_{k+1})(k+1)K_k(x) + (n+1)(\Delta a_n - \Delta a_{n+2})K_n(x) + (a_n - a_{n+2})K_n(x)$

Thus

$||f - g_n|| \le \int_0^{\pi} |\sum_{k=n+1}^{\infty}\Delta^2(a_{k-1} - a_{k+1})(k+1)K_k(x)|dx + (n+1)|\Delta a_n - \Delta a_{n+2}|\int_0^{\pi}|K_n(x)|dx + |a_n - a_{n+2}|\int_0^{\pi}|K_n(x)|dx$ .... (2.2)

Now

$$|\Delta a_n - \Delta a_{n+2}| = \left|\sum_{k=n}^{\infty}(\Delta^2 a_k - \Delta^2 a_{k+2})\right|$$
$$= \left|\sum_{k=n+1}^{\infty}\frac{k}{k}(\Delta^2 a_{k-1} - \Delta^2 a_{k+2})\right|$$
$$= \frac{1}{n+1}\left|\sum_{k=n+1}^{\infty}(\Delta^2 a_{k-1} - \Delta^2 a_{k+1})\right|$$
$$= o\left(\frac{1}{n+1}\right) \text{........(2.3)}$$

and also we know

$\int_0^{\pi}|K_n(x)|dx \le \int_{-\pi}^{\pi}|K_n(x)|dx = \pi$ ....(2.4)

Considering (2.3), (2.4) and our assumption that sequence {a_n} belongs to class K, we have from (2.2)

$$||f - g_n|| = o(1), n \to \infty$$

Note: The result of above Theorem 1 can also be prvoed for the generalised class K^{α} by applying Abel's transformation α times to (2.1)

Corollary. Let $\{a_k\}$ belongs to class K, then the necessary and sufficient condition for L^1-convergence of the cosine series (1.1) is

$|a_n - a_{n+2}| \log n = o(1), n \to \infty$

Proof. We noticed that

$$\int_{-\pi}^{\pi} |f(x) - S_n(x)| dx$$

$$\leq \int_{-\pi}^{\pi} |f(x) - g_n(x)| dx + \int_{-\pi}^{\pi} |g_n(x) - S_n(x)| dx$$

$$\leq \int_{-\pi}^{\pi} |f(x) - g_n(x)| dx + \int_{-\pi}^{\pi} \left|\frac{a_n - a_{n+2}}{n+1} \tilde{D}'_n(x)\right| dx \text{ ....(1)}$$

Since $\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - g_n(x)| dx = 0$ by our theorem and

$$\int_{-\pi}^{\pi} \left|\frac{a_n - a_{n+2}}{n+1} \tilde{D}'_n(x)\right| dx \sim |a_n - a_{n+2}| \log n \text{ as } \int_{-\pi}^{\pi} \left|\tilde{D}'_n(x)\right| dx \sim n \log n (n \geq 2)$$

Thus form (1). We have $||f - S_n|| = o(1)$ as $n \to \infty$

Conversely, we have

$$\int_{-\pi}^{\pi} \left|\frac{a_n - a_{n+2}}{n+1} \tilde{D}'_n(x)\right| dx = \|g_n - S_n\|$$

$\leq ||f - g_n|| + ||f - S_n||$

Now as $||f - g_n|| = o(1), n \to \infty$ by our Theorem 1 and

$||f - S_n|| = o(1), n \to \infty$ is given

Hence $||f - S_n|| = o(1), n \to \infty$ iff $\lim_{n\to\infty} |a_n - a_{n+2}| \log n = 0$

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